Monetary Policy and Wealth Effects*

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Abstract

We study the role of wealth effects in New Keynesian models. We extend the Slutsky decomposition to a general equilibrium setting. Wealth effects, and their amplification in general equilibrium, explain a large fraction of the consumption and inflation response in the standard equilibrium. In RANK, wealth effects are determined, generically, by the response of fiscal variables to monetary policy. The estimated fiscal response is several times smaller than the implied fiscal response in our DSGE model. In a HANK model with private debt, private wealth effects amplify the response to monetary policy, improving the quantitative performance of the DSGE model.

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1 Introduction

There is a long tradition in monetary theory that emphasizes the role of wealth effects in the monetary policy transmission mechanism. The importance of wealth effects, i.e., the revaluation of financial and human wealth in response to changes in monetary policy, can be traced back to classical economists, such as Pigou, as well as Keynesian economists, such as Patinkin, Metzler, and Tobin.[1] Keynes himself described in the General Theory the effect of interest rate changes as follows:[2]

> There are not many people who will alter their way of living because the rate of interest has fallen from 5 to 4 per cent, if their aggregate income is the same as before. [...] Perhaps the most important influence, operating through changes in the rate of interest, on the readiness to spend out of a given income, depends on the effect of these changes on the appreciation or depreciation in the price of securities and other assets.

Even though wealth effects continued to permeate much of the monetary analysis following Keynes, little is known about the importance of this channel in our modern frameworks. In this paper, we provide a reassessment of such wealth effects in modern New Keynesian models.

The paper derives three main results. First, in the context of a simple Representative Agent New Keynesian (RANK) model, we propose a decomposition of the equilibrium response of consumption to changes in nominal interest rates into three components: a (general equilibrium) substitution effect, a wealth effect, and an interaction term capturing the amplification of wealth effects. We find that a large fraction of the consumption response is driven by wealth effects and their amplification term. Moreover, the initial response of inflation is entirely driven by wealth effects, a result that sheds new light on the mechanisms through which the central bank controls inflation in these models.

Second, we assess the quantitative importance of wealth effects in a medium-scale DSGE model along the lines of Christiano et al. (2005). We find that, in the absence of

[1] See e.g. Pigou (1943), Metzler (1951), Patinkin (1965), and Tobin (1969).

[2] Book III, Chapter 8, Section II, of "The General Theory of Employment, Interest, and Money."

wealth effects, the DSGE model is unable to match the dynamics of consumption observed in the data. This illustrates how the well-known quantitative success of such DSGE models relies on their ability to generate sufficiently strong wealth effects. In RANK models, these effects depend, generically, on the revaluation of public debt and the response of fiscal policy. We estimate such fiscal-based wealth effects in the data and find that they are several times smaller than what is required by the model to match the observed consumption dynamics. Moreover, when we require the model to match the response of the value of public debt and of the fiscal transfers observed in the data, the quantitative performance of the DSGE model deteriorates significantly.

Because public assets cannot generate the required level of wealth effects, we next consider the role of private assets. Private assets can affect the transmission mechanism of monetary policy only if agents have heterogeneous marginal propensities to consume (MPC), as otherwise any gain or loss to savers due to changes in interest rates would be exactly offset by a corresponding loss or gain to borrowers, with no effect on aggregates. Therefore, in our third main result, we introduce both heterogeneity and private assets in the model and show that private wealth effects substantially amplify the response to monetary policy. Our focus on the interaction of heterogeneous MPCs with a positive amount of private debt contrasts with much of the literature on analytical HANK models, which typically focuses on the case of zero liquidity.[3] Furthermore, we show that introducing private wealth effects significantly improves the ability of our quantitative model to match the empirical response of consumption to monetary shocks, even when the model is required to match the response of fiscal variables observed in the data.

We now consider each one of these results in detail. We begin by presenting a decomposition of the response of the economy to changes in the path of the nominal interest rate in the context of a standard RANK model in continuous time, in the spirit of Werning (2012) and Cochrane (2017). Our decomposition consists of an extension of the standard Slutsky decomposition to a general equilibrium setting. In partial equilibrium, a change

[3] See e.g., Broer et al. (2019) and Bilbiie (2018) for analytical HANK models with zero private liquidity. Our result is closer to the one in Auclert (2019), who discusses the interaction of heterogeneous MPCs with differences in interest rate exposures. However, while he presents a consumption decomposition in terms of real interest rate and output, our results are expressed directly in terms of policy variables, namely nominal interest rates and fiscal transfers.

in relative prices triggers a change in the households' consumption decisions through two basic channels: a substitution effect and a wealth effect. In general equilibrium, however, the wealth effect would affect inflation, changing intertemporal prices and affecting the substitution effect. Therefore, in general equilibrium, the substitution and wealth effects are intertwined, and it is important to explicitly take into account their interaction.

Our first main analytical result shows that, given a change in the path of the nominal interest rate, the path of consumption in *all* solutions to the system given by the Euler equation and the New Keynesian Phillips Curve can be expressed as the sum of three components: a *general equilibrium substitution effect (GE-SE)*, which accounts for the change in the consumption and inflation paths generated by the change in nominal interest rates, keeping households' wealth fixed; a *wealth effect (WE)*, which accounts for changes in the present value of the households' labor and financial income, as well as changes in the present value of fiscal transfers and in the value of public debt; and a *general equilibrium (GE) amplification*, given by the product of the wealth effect and a *GE multiplier*, which incorporates the general equilibrium interactions between price determination and households' wealth. Moreover, this decomposition has a useful alternative interpretation. The GE-SE is the *unique* solution to the RANK model in which households' wealth does not change; the WE and the GE amplification, then, incorporate the changes in the economy triggered by variations in households' wealth.

The decomposition provides new insights into several dimensions of the economic mechanisms embedded in New Keynesian models. First, we identify a new interpretation of the source of the multiplicity of equilibria that plagues the RANK model with an interest rate peg. We show that the GE multiplier is uniquely determined by the structural parameters of the model, and the GE-SE is unique given a path of the nominal interest rate. Thus, we can index all solutions to the RANK model by the level of wealth effect they generate. In this sense, we can interpret the Taylor rule as selecting a particular level of the equilibrium wealth effect. In particular, we show that, under standard parametrizations, the Taylor rule equilibrium selects the unique purely forward-looking solution of the system. Moreover, the Fiscal Theory of the Price Level (FTPL), in the version with sticky prices, differs from the equilibrium with a Taylor rule only in the determination of the wealth effect. Therefore, our approach can handle both active monetary and active fiscal policy equilibria in an unified way.[4]

Second, the interaction of the WE with the general equilibrium determination of prices may be quantitatively important, even when the WE itself has only a minor effect on equilibrium consumption. The GE multiplier is positive when prices are sticky and can be quantitatively large. For a standard calibration of the simple model, the WE is amplified by more than twenty times in general equilibrium. In the case of the Taylor rule equilibrium, for instance, we find that the WE is quantitatively small, but the GE amplification, the product of the WE and GE multiplier, accounts for 55% of the total consumption response.[5]

Third, we show that a similar decomposition holds for inflation. Interestingly, we find that initial inflation is proportional to the wealth effect and is not affected by the GE-SE. That is, what generates a drop in current inflation is the fact that *average* consumption decreases, rather than the change in the timing of consumption. This result might have important consequences for policy design, since it implies that, according to RANK, policy needs to make households poorer in order to lower inflation, while changing the timing of their consumption is irrelevant to the short-run response of inflation.

Fourth, we show that if monetary policy has fiscal consequences (that is, if it affects either tax revenues or the cost of servicing public debt) then the wealth effect is proportional to the sum of the change in the value of public bonds and the present value of government transfers, after the endogenous response of profits and wages are taken into account. This result provides a one-to-one mapping between the level of wealth effects in RANK models and the response of fiscal variables, and it provides a novel testable implication.

Next, we take the implications of the decomposition to the data and quantify the importance of wealth effects. We do this in steps. First, we estimate the impulse-response functions (IRFs) to a monetary shock using the standard recursiveness assumption proposed by Christiano et al. (2005).[6] The innovation of our approach is that we include fiscal variables in the VAR system to capture the fiscal response to a monetary shock. By doing this, we

[4] For a discussion of policy regimes and FTPL, see Sims (1994), Woodford (1995), and Cochrane (2001).

[5] This is consistent with the finding by Kaplan et al. (2018) that the indirect effect, which is related to the WE, is quantitatively small in the purely forward-looking solution to the New Keynesian model. For the relationship between direct/indirect effects and substitution/wealth effects, see Section 2.5.

[6] See Christiano et al. (1999) for a detailed exposition of this strategy.

obtain an estimate of the counterpart of the lump-sum transfers in the model. Second, we estimate a cashless version of Christiano et al. (2005) by impulse-response matching. Importantly, we estimate the model assuming that monetary policy follows a standard Taylor rule. In this sense, our estimated model belongs to the strand of literature that assumes an active-monetary/passive-fiscal regime. Our estimation generates parameter values that are in line with those obtained in Christiano et al. (2005) and impulse responses that are roughly consistent with those estimated in the data. We then extend our analytical decomposition to the DSGE model and show that WE and its amplification in general equilibrium accounts for more than 80% of the consumption response in our estimated model. Therefore, to match the empirical IRFs, it is crucial that the model generates a sufficient level of wealth effect. Finally, using a calibrated version of the government's budget constraint, we are able to recover the implied fiscal response that is necessary to sustain the consumption level predicted by the model.

With all these elements, we are able to evaluate the quantitative performance of the model. First, we show that the fiscal response we obtain from the data is (statistically) significantly lower than the one implied by the model. That is, to generate impulse responses that are close to their counterparts in the data, the model requires a fiscal response that is higher than the one estimated in the data. Moreover, the magnitude of the fiscal response in the model is almost 5 times larger than the one obtained in the data. Given that the vast majority of the consumption response in the model can be accounted for by the GE amplification, this result suggests that the quantitative performance of the model relies on a *counterfactual* fiscal response.

The importance of these fiscal-based wealth effects can be more clearly seen by considering the following exercise. We plug into the model the estimated impulse response functions of policy variables, i.e., nominal interest rate, taxes, and transfers. That is, we drop the Taylor rule and force the model to *exactly* match the IRFs for policy variables estimated in the data. This way, the wealth effect generated by the model will be, by construction, consistent with the response of fiscal variables in the data. We then calculate the equilibrium of the economy. We find a dramatic change in the model's predictions, as it now *fails* to match the data. For instance, while initially the model was generating a recession in response to a contractionary monetary policy shock, it now generates a *boom*, inconsistent with the pattern observed empirically. By imposing the constraint that the implied fiscal response is consistent with the data, the initial success of the model turned into a severe failure. This result illustrates how considering the different mechanisms through which monetary policy affects the economy, instead of simply considering monetary policy's overall effect, can be useful in disciplining the theory empirically.

We next consider the role of private wealth effects in a borrower-saver economy. Savers are unconstrained in equilibrium, while borrowers consume their income net of interest payments on the debt. Importantly, nominal interest rates now affect aggregate consumption directly through their impact on the cost of servicing private debt. We start by providing a characterization of the aggregate dynamics of the economy and show that aggregate consumption satisfies a generalized Euler equation. We then extend our decomposition to the economy with heterogeneous agents. Wealth effects can now be decomposed into two components: an average wealth effect, which is determined by fiscal variables as in the RANK model, and a *private wealth effect*, which depends on the amount of private debt and on the path of nominal interest rates. Private wealth effects substantially amplify the impact of monetary policy. Compared with an economy with zero liquidity, we find that introducing private debt raises the initial response of consumption to changes in nominal interest rates by 50% in a calibrated example. Moreover, private wealth effects now account for 53% of the total consumption response. Finally, we introduce heterogeneous agents and private assets in our quantitative DSGE model. We maintain the constraint that the model must match the response of the fiscal variables observed in the data. The ability of the model to match the dynamics of consumption improves substantially, generating a decline in consumption in response to a monetary tightening in line with the decline estimated in the data.

Literature review. Our emphasis on wealth effects reflects a long tradition in monetary economics. Pigou (1943) relied on a wealth effect, through the revaluation of money balances, to argue that full employment could be reached even in a liquidity trap. Kalecki (1944) pointed out that these effects apply only to government liabilities, as the impacts of inside assets cancel out in the aggregate, implicitly assuming no heterogeneity in MPCs. The Pigou effect is then consistent with our formulation of the FTPL, which also focus on government bonds.[7] Keynesian economists criticized the Pigou effect for being likely small empirically, consistent with our empirical results on fiscal-based wealth effects. Tobin (1982) explicitly defended the idea that private debt was likely much more important in an economy with heterogeneous MPCs, echoing our results on private wealth effects.

Our paper is related to the literature studying the transmission mechanism of monetary policy in RANK and HANK models.[8] Kaplan et al. (2018) proposes a decomposition in terms of direct and indirect effects, which we discuss in detail in Section 2.5. Auclert (2019) decomposes the response of consumption into substitution and wealth effects in an economy with rich heterogeneity. In contrast to our results, his decomposition is expressed in terms of aggregate output and the real interest rate. Our decomposition instead is in terms of policy variables, i.e. nominal interest rates and fiscal transfers. Broer et al. (2019) and Rupert and Šustek (2019) also focused on the transmission mechanism of New Keynesian models, but they do not emphasize the role of substitution and wealth effects.

An important distinction between the analysis in this paper and much of the HANK literature is our focus on *macro* channels of monetary policy transmission rather than their *micro* counterpart. Micro channels refer to economic forces that operate at the individual level, while macro channels do so on aggregate variables. As Werning (2015) shows, one can construct HANK models in which the monetary policy transmission at the individual level (i.e., the micro channels) is substantially different from that in the RANK (emphasizing the importance of income effects by breaking the permanent income hypothesis), while at the same time aggregate variables behave *as-if* they were generated by a representative agent, thus generating no change at the aggregate level (i.e., the macro channels). In our HANK model, we emphasize the interaction between heterogeneous MPCs and positive debt in generating stronger *macro* channels that generate quantitative results closer to the data.

[7] For a discussion of the Pigou effect, see Patinkin (1948). The importance of wealth effects from government bonds were questioned by Barro (1974) in the context of a real economy. While in real economies any changes in the value of government bonds must necessarily be offset by changes in taxes, this is not the case in monetary economies. See e.g. Woodford (1998).

[8] See also a recent literature on housing wealth effects, e.g., Berger et al. (2017) and Guren et al. (2018).

Our work is also related to the literature on analytical HANK models, such as Werning (2015), McKay et al. (2017), Debortoli and Galí (2017), Acharya and Dogra (2018), Bilbiie (2018), and Bilbiie (2019). Much of the literature focuses on the case of zero private liquidity and idiosyncratic income risk, while we focus on the complementary case of positive private liquidity and no idiosyncratic income risk.[9] The presence of private debt generates a *compounding* Euler equation, i.e., current consumption responds more strongly to future interest rate changes than with the standard Euler equation. The emphasis on private debt is also shared by Eggertsson and Krugman (2012) and Benigno et al. (2019).

Finally, our paper is related to the literature on monetary-fiscal interactions; see e.g., Cochrane (2017) and Cochrane (2019) and the review by Leeper and Leith (2016). Consistent with our results, the intuition behind FTPL models is usually expressed in terms of wealth effects. Our results, however, do not rely on a particular fiscal rule and hold for either passive or active monetary regimes.

2 A General Equilibrium Decomposition of Consumption

In this section, we consider a simple RANK in continuous time. The model is based on Werning (2012) and Cochrane (2017), augmented to incorporate fiscal variables and explicitly account for the households' budget constraint. The main result of this section presents a decomposition of the response of consumption to a monetary policy shock that is an extension of the Slutsky equation to a general equilibrium environment. We then use this decomposition to quantify the importance of wealth effects in the general equilibrium of the economy.

We study the dynamic response of an economy that is hit by a monetary shock, resulting in a deviation of the path of nominal interest rates from its steady-state level and a simultaneous response of the fiscal authority. We analyze the reaction of the economy to the resulting equilibrium paths of the *nominal interest rate* and *fiscal variables*. By focusing the analysis on the equilibrium paths of policy variables, we are able to obtain results that are robust to any monetary/fiscal regime that generates such paths.

[9] Acharya and Dogra (2018) allows for positive private liquidity, but abstracts from heterogeneous MPCs.

2.1 The Model

Environment. Time is continuous and denoted by $t \in \mathbb{R}_+$. The economy is populated by a large number of identical, infinitely-lived households and a government. There is also a continuum of mass one of firms that produce a differentiated good using labor as the only factor of production. Households' preferences are such that consumption is a CES aggregator of the purchases of each of the differentiated goods. The government chooses the path of the nominal interest rate, levies proportional sales taxes, issues short-term nominal debt (which is in positive net supply in steady state) and distributes lump-sum transfers (which are allowed to be negative).[10] As is standard in the literature, we log-linearize the model around its zero inflation steady-state equilibrium and consider the first-order approximation of the response to exogenous shocks.

Given a path of interest rates $\{i_t\}_{t=0}^{\infty}$ and transfers $\{T_t\}_{t=0}^{\infty}$, the log-linearized solution of the model can be characterized by four equations: an intertemporal Euler equation

$$\dot{c}_t = \sigma^{-1}(i_t - \pi_t - \rho), \tag{1}$$

a New Keynesian Phillips Curve

$$\dot{\pi}_t = \rho \pi_t - \kappa c_t, \tag{2}$$

the households' intertemporal budget constraint

$$\int_0^\infty e^{-\rho t} c_t dt = \int_0^\infty e^{-\rho t} \left[(1-\tau) y_t + b(i_t - \pi_t - \rho) + T_t\right] dt, \tag{3}$$

and the resource constraint

$$c_t = y_t, \tag{4}$$

where c_t and y_t denote, respectively, the percentage difference between actual consumption and output and their corresponding levels in a steady state, c and y; π_t denotes the inflation rate; i_t denotes the nominal, short-term, risk-free interest rate; σ denotes the inverse of

[10] In this paper, we abstract from government spending. Given our focus on monetary policy shocks, we follow the literature and assume that there is no response of government spending to changes in monetary policy.

the intertemporal elasticity of substitution; ρ denotes the households' subjective discount factor; κ is the slope of the Phillips curve; τ is the steady-state rate of proportional sales taxes; and b is the steady-state level of short-term nominal debt.

Since our analysis emphasizes the role of the households' budget constraint in the dynamic behavior of consumption, it is useful to briefly describe its components. The left-hand side of equation (3) is the present value of consumption, discounted at the steady-state real interest rate. The right-hand side contains the sources of income: the after-tax profits and wages, which combined equal $(1-\tau)y_t$, the interest from financial assets, and government's lump-sum transfers.[11] There are two channels through which fiscal variables affect the budget constraint of the households. First, they affect non-interest income through τ and T_t. Second, the level of government debt determines the households' exposure to real interest rate changes. While changes in the real interest rate affect the present discounted value of both consumption and after-tax income, in a representative-agent economy the net impact depends only on the steady-state level of government debt.[12]

Role of policy rules. As mentioned above, our exercise focuses on the paths of policy variables and studies the channels through which these paths affect equilibrium dynamics. This exercise differs from the standard approach in the literature, which typically assumes monetary and fiscal rules and then determines the equilibrium path of policy variables endogenously. A popular approach is to assume that monetary policy follows an interest rate rule of the form

$$i_t = \rho + \phi \pi_t + \varepsilon_t, \tag{5}$$

where $\phi > 1$ and ε_t represents an innovation of the rule relative to its systematic response to inflation. Fiscal policy is assumed to be *passive* or *Ricardian*, and the exogenous monetary shock is represented by a path for $\{\varepsilon_t\}_{t=0}^{\infty}$ rather than a path for the nominal interest

[11]We have assumed that the proportional taxes $\{\tau_t\}_{t=0}^{\infty}$ are fixed at their steady-state level. Since changes in proportional taxes are not a channel emphasized in the monetary policy literature, we abstract from it in this paper.

[12]Formally, the impact of changes in the interest rate on the present discounted value of consumption is $-\frac{c}{\rho}\int_0^\infty e^{-\rho t}(i_t - \pi_t - \rho)dt$, and the corresponding impact on after-tax income is $-\frac{(1-\tau)y+T^*}{\rho}\int_0^\infty e^{-\rho t}(i_t - \pi_t - \rho)dt$, T^* being the steady-state transfers. Combining the two and using $c = (1-\tau)y + T^* + \rho b$, we obtain $\int_0^\infty e^{-\rho t} b(i_t - \pi_t - \rho)dt$.

rate.[13] Under these assumptions, equation (3) is often dropped when finding an equilibrium of the economy because transfers $\{T_t\}_{t=0}^{\infty}$ are assumed to automatically adjust so that the government's budget constraint is satisfied for any path of the endogenous and exogenous variables.[14] Since lump-sum transfers do not affect any of the other equations characterizing equilibrium, they represent a free variable that adjusts to guarantee that any solution to the system given by (1), (2) and (5) is an equilibrium of the economy.

An alternative approach would be to follow the Fiscal Theory of the Price Level (FTPL) to specify an exogenous path for the fiscal transfers $\{T_t\}_{t=0}^{\infty}$ and assume an interest rule (5) with $\phi < 1$, corresponding to an *active* fiscal regime. Despite the stark differences between the two approaches, our formulation is consistent with both. The determination of the paths of policy variables, $\{i_t\}_{t=0}^{\infty}$ and $\{T_t\}_{t=0}^{\infty}$, depends on the specific monetary/fiscal regime in place. By analyzing the impact of the policy variables directly on consumption and inflation, we are able to bypass the debate on the correct monetary/fiscal policy regime and obtain results about the monetary policy transmission channels that are robust to different regimes.

Dynamic system. The system of differential equations (1)-(2) can be written as

$$\begin{bmatrix} \dot{c}_t \\ \dot{\pi}_t \end{bmatrix} = \begin{bmatrix} 0 & -\sigma^{-1} \\ -\kappa & \rho \end{bmatrix} \begin{bmatrix} c_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma^{-1}(i_t - \rho) \\ 0 \end{bmatrix}.$$

The eigenvalues of the system above are given by

$$\overline{\omega} = \frac{\rho + \sqrt{\rho^2 + 4\sigma^{-1}\kappa}}{2} > 0, \quad \underline{\omega} = \frac{\rho - \sqrt{\rho^2 + 4\sigma^{-1}\kappa}}{2} < 0.$$

Note that the system has one positive and one negative eigenvalue. Focusing on bounded solutions, we need one additional condition to determine equilibrium. We show below that, generically, knowledge of $\{i_t\}_{t=0}^{\infty}$ and $\{T_t\}_{t=0}^{\infty}$ is enough to pin down a solution.

[13] Note that Ricardian equivalence holds in this model regardless of the monetary/fiscal regime, so only the present value of transfers, $\int_{t=0}^{\infty} e^{-\rho t} T_t dt$, rather than the whole path, $\{T_t\}_{t=0}^{\infty}$, matters for the equilibrium.

[14] See e.g., Woodford (2011) for a discussion.

2.2 Consumption Decomposition: Substitution and Wealth Effects

We consider next the different channels through which changes in nominal interest rates affect consumption. The main result of this section decomposes consumption into three terms: a *general-equilibrium substitution effect (GE-SE)*, a *wealth effect (WE)*, and a *general-equilibrium (GE) amplification*. Moreover, we show that the multiplicity of solutions that plagues the New Keynesian system can be indexed by the amount of WE each solution generates. This exercise provides a formal characterization of the monetary policy transmission mechanism in the standard RANK model.

Substitution and wealth effects. We begin by defining two objects that represent the core of the characterization that follows. First, for a given path of the nominal interest rate and inflation, $\{i_t, \pi_t\}_{t=0}^{\infty}$, the households' *Hicksian demand* is given by

$$c_t^H \equiv \sigma^{-1} \int_0^t (i_s - \pi_s - \rho) ds - \sigma^{-1} \int_0^\infty e^{-\rho s} (i_s - \pi_s - \rho) ds. \tag{6}$$

Equation (6) is the log-linear approximation of the solution to the minimization of a household's expenditures subject to achieving at least the steady-state level of utility.[15] In this setting, the different goods are consumption at different dates, and the price of one unit of consumption at date t is $e^{-\int_0^t (i_s - \pi_s) ds}$. An important property of the Hicksian demand is that the total cost of the bundle $\{c_t^H\}_{t=0}^{\infty}$ evaluated at steady-state prices is zero, that is

$$\int_0^\infty e^{-\rho t} c_t^H dt = 0. \tag{7}$$

The Hicksian demand will be tightly connected to the substitution effect in general equilibrium.

The second object is the average consumption, which is given by

$$C \equiv \rho \int_0^\infty e^{-\rho t} c_t dt = \rho \int_0^\infty e^{-\rho t} \left[(1-\tau) y_t + b(i_t - \pi_t - \rho) + T_t \right] dt. \tag{8}$$

Average consumption is the consumption path that would prevail if the households were

[15] See Appendix B for the details of the derivation.

forced to a constant consumption path while still satisfying their budget constraint.

From the standard properties of a Marshallian demand system, consumption can be written as

$$c_t = c_t^H + C. \tag{9}$$

The expression above corresponds to a (log-linear) version of the Slutsky decomposition of consumption, extended to allow for simultaneous changes in prices and income in all periods. In partial equilibrium, the first component corresponds to the substitution effect, which captures the impact of interest rate changes on the *timing* of consumption, while the second component corresponds to the wealth effect, which captures the effect on the *level* of consumption. In general equilibrium, however, these two terms are not independent of each other. A positive wealth effect, for instance, would generate inflation and change intertemporal prices, affecting the timing of consumption and the substitution effect. To disentangle the two effects, we propose the following definition of the *general-equilibrium substitution effect*.

Definition 1 (General-Equilibrium Substitution Effect (GE-SE))**.** *The* general-equilibrium substitution effect *is the Hicksian demand evaluated at the equilibrium path of nominal rates,* $\{i_t\}_{t=0}^{\infty}$*, and the inflation* induced *by the substitution effect,* $\{\pi_t^S\}_{t=0}^{\infty}$*. That is,* $\{c_t^S, \pi_t^S\}_{t=0}^{\infty}$ *is the solution to the following system of equations*

$$c_t^S = \sigma^{-1} \int_0^t (i_s - \pi_s^S - \rho) ds - \sigma^{-1} \int_0^\infty e^{-\rho s} (i_s - \pi_s^S - \rho) ds, \tag{10}$$

$$\pi_t^S = \kappa \int_t^\infty e^{-\rho(s-t)} c_s^S ds. \tag{11}$$

where (10) *is the Hicksian demand evaluated at* $\{i_t, \pi_t^S\}_{t=0}^{\infty}$ *and* (11) *is the New Keynesian Philips curve integrated forward.*

The GE-SE corresponds to the solution to the fixed-point problem given by the Hicksian demand evaluated at a path of inflation that is itself consistent with the Hicksian demand. Definition 1 allows the substitution effect to incorporate the feedback between the households' decisions and the general equilibrium determination of prices in the economy, but only through interactions arising from the dynamics of the Hicksian demand. Moreover, the GE-SE has properties that connect it to the original RANK system given by (1)-(2). Since the consumption bundle prescribed by the Hicksian demand has zero cost (relative to the steady-state bundle; see equation (7)), the GE-SE is the solution to the system of equations (1)-(2), imposing no change in the households' wealth.

Lemma 1. *The solution to* (10)-(11) *is the* unique *solution to the system given by* (1)-(2) *and*

$$\int_0^\infty e^{-\rho t} c_t dt = 0.$$

Thus, we could have alternatively defined the GE-SE as the solution to the RANK system of differential equations that keeps the households' wealth unchanged. More importantly, Lemma 1 establishes that the GE-SE generates a *unique* path for the Hicksian demand.

Next, we present the main result of this section. Proposition 1 shows how to decompose equilibrium consumption into substitution and wealth effects that resemble the standard decomposition in consumer theory, augmented to account for general equilibrium effects.

Proposition 1 (Consumption Decomposition in General Equilibrium)**.** *Given an equilibrium path for the nominal interest rate, $\{i_t\}_{t=0}^{\infty}$, all bounded solutions to the system* (1)-(2) *generate a path of consumption that can be written as*

$$c_t = \underbrace{c_t^S}_{\text{GE-SE}} + \underbrace{C}_{\text{wealth effect}} + \underbrace{\overbrace{\left(\frac{\overline{\omega}}{\rho} e^{\underline{\omega} t} - 1\right)}^{\text{GE multiplier}} \times C}_{\text{GE amplification}},$$

where c_t^S is given by (10)*, and C is given by* (8)*.*

In $t = 0$, the GE multiplier is weakly positive, and strictly so if $\kappa > 0$.

The equilibrium response of consumption to a monetary shock can be decomposed into three terms. The first term is the GE-SE. A change in the nominal interest rate represents a change in the relative price of consumption today relative to tomorrow. A household's response to this change corresponds to the substitution effect: an increase in interest rates leads the household to shift consumption from the present to the future, while keeping

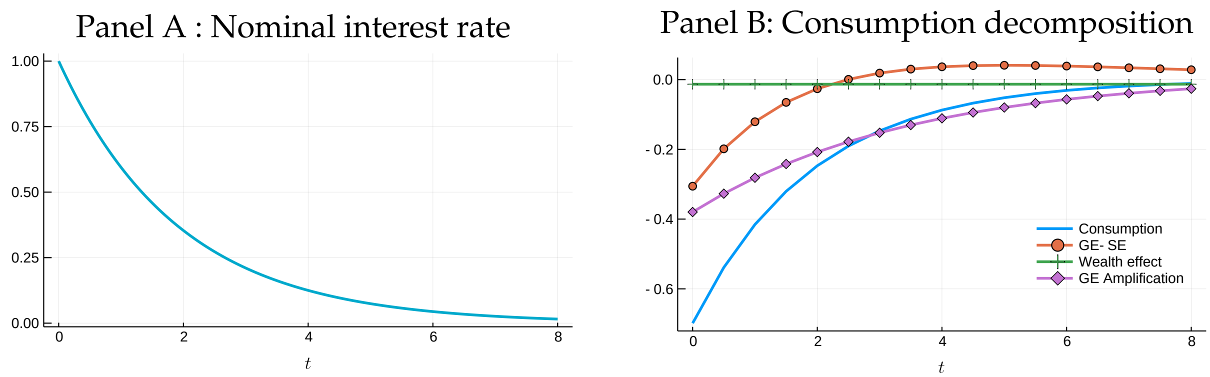

FIGURE 1: Decomposition of the consumption response to a nominal interest rate change
Calibration: $\sigma = 1$, $\kappa = 0.09$, $\rho = 0.01$. Half-life of nominal interest rate is four months. The solution corresponds to the unique forward-looking equilibrium.

the total cost of the bundle fixed. The GE-SE incorporates the effects that the Hicksian demand has on the determination of prices, as specified by the New Keynesian Phillips curve (11). It is, thus, in this sense a general-equilibrium substitution effect. The second term is the wealth effect. The change in the path of the nominal interest rate impacts the household's average consumption, as the change in interest rates implies a revaluation of the household's after-tax financial and human wealth. The third term shows that the wealth effect can be amplified in general equilibrium. When the household feels richer and increases its consumption, it puts upward pressure on inflation. For a given equilibrium path of the nominal interest rate, the increase in inflation reduces the real interest rate, further stimulating the economy. This effect is captured by the product of the wealth effect and a term that depends only on parameters of the model, which we call the *GE multiplier*.

In order to determine the quantitative importance of each component, we present a numerical example in Figure 1. The solid lines represent the equilibrium paths of the nominal interest rate (Panel A) and the household's consumption (Panel B). We depicted a standard equilibrium in which an increase in the nominal interest rate generates a reduction in the path of consumption.[16] Panel B also decomposes the equilibrium response of consumption into the components defined in Proposition 1. All components of consumption are negative on impact. In terms of their contribution to the total response, we find that the GE-SE accounts for 43% of the total response, the wealth effect accounts for 2% and the GE ampli-

[16]In Figure 1, we focus on the unique forward-looking solution to the system (1)-(2), which coincides with the standard Taylor equilibrium, as shown in Lemma 2 below.

fication accounts for 55%. That is, less than half of the total response of consumption can be attributed to the *intertemporal substitution channel*, while the direct role of the wealth effect is marginal. The small impact of the wealth effect on equilibrium consumption is consistent with the fact that the representative household in the model conforms to the permanent income hypothesis: changes in the household's wealth get smoothed out over time, and their impact on any given period is proportional to the discount rate ρ, which is typically small. However, the wealth effect gets magnified by the GE multiplier, to the point that the GE amplification accounts for more than half of the total response. That is, even in the RANK model, the wealth effect plays a substantial role, though indirectly, through powerful endogenous amplification mechanisms. Section 3 revisits this decomposition in a medium-scale DSGE version of the model.

Next, we provide some insights about the source of multiplicity in the New Keynesian model.

Corollary 1.1. *Given a path for the nominal interest rate, $\{i_t\}_{t=0}^{\infty}$, all bounded solutions to the system* (1)-(2) *generate the same GE-SE and GE multiplier.*

The decomposition in Proposition 1 characterizes *all* the bounded solutions to the system (1)-(2) for a given path of the nominal interest rate. Corollary 1.1 establishes that all these solutions produce the same GE-SE and GE multiplier. This result provides a new perspective on the multiplicity of equilibria of the New Keynesian model under an interest rate peg. Corollary 1.1 implies that *all* solutions of the New Keynesian model under an interest rate peg can be indexed by their effect on average consumption, that is, by the level of wealth effect they generate. In this sense, the standard Taylor rule equilibrium and the FTPL are ways of selecting a particular level of wealth effect.[17] We next consider the Taylor rule equilibrium in detail.

Wealth effects in the Taylor rule equilibrium. Consider the interest rate rule (5) with $\phi \in \left[1, 1 + \frac{\rho^2}{4\kappa\sigma^{-1}}\right)$.[18] We say that a sequence of monetary shocks $\{\varepsilon_t\}_{t=0}^{\infty}$ *decays sufficiently*

[17]This interpretation is valid conditional on these rules producing the same equilibrium path for the nominal interest rate. In more general settings, the two rules could potentially have different implications for the equilibrium path of the interest rate, and hence, for the decomposition.

[18]We restrict the values of ϕ in order to obtain real valued eigenvalues.

fast if $\varepsilon_t = \mathcal{O}(e^{-\theta t})$, where $\theta > |\underline{\omega}|$. Under this assumption, the Taylor rule equilibrium is the unique purely forward-looking solution of the New Keynesian system.[19]

Lemma 2. *Suppose the equilibrium path of the nominal interest rate, $\{i_t\}_{t=0}^{\infty}$, was generated by an interest rule that satisfies the Taylor principle, given a sequence of shocks $\{\varepsilon_t\}_{t=0}^{\infty}$ that decays sufficiently fast. Then, the equilibrium path of consumption is the unique purely forward-looking solution to the system* (1)-(2), *that is,*

$$c_t = -\frac{\sigma^{-1}}{\overline{\omega} - \underline{\omega}} \int_t^{\infty} \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)} \right) (i_s - \rho)\, ds.$$

The corresponding wealth effect is

$$C = -\frac{\sigma^{-1}\rho}{\overline{\omega} - \underline{\omega}} \int_0^{\infty} \left(e^{-\underline{\omega} t} - e^{-\overline{\omega} t} \right) (i_t - \rho)\, dt.$$

Lemma 2 shows how consumption responds to changes in the nominal interest rate in the Taylor rule equilibrium.[20] Two features of the solution are particularly relevant. First, the Taylor rule solution corresponds to the *unique* purely forward-looking solution to the system (1)-(2). Note that the substitution effect on date t depends on both past and future interest rates. Moreover, the wealth effect can depend, in principle, on the entire path of nominal interest rates. Therefore, the solution to the system (1)-(2) has, in general, both backward-looking and forward-looking components. There is a unique value of C, however, such that the effect of past interest rates on the substitution effect and on the sum of the wealth effect and GE amplification cancel out exactly, and this corresponds to the Taylor rule solution. Second, an increase in nominal interest rates leads to a decline in consumption on all dates, as can be seen also in Figure 2. An increase in interest rates implies, then, a negative wealth effect under a Taylor rule.

The importance of wealth effects for the Taylor rule equilibrium can be seen by comparing it with the GE-SE. As can be seen in Panel A of Figure 2, the same path of interest rates now generates a decline in consumption in the first year and a small boom afterwards, such

[19] The condition $\theta > |\underline{\omega}|$ guarantees that a positive monetary shock leads to an increase in the nominal interest rate, as in standard calibrations of the New Keynesian model.

[20] The nominal interest rate is, of course, endogenous under a Taylor rule. The expression in the lemma is to be interpreted as a restriction on the joint behavior of consumption and nominal interest rates.

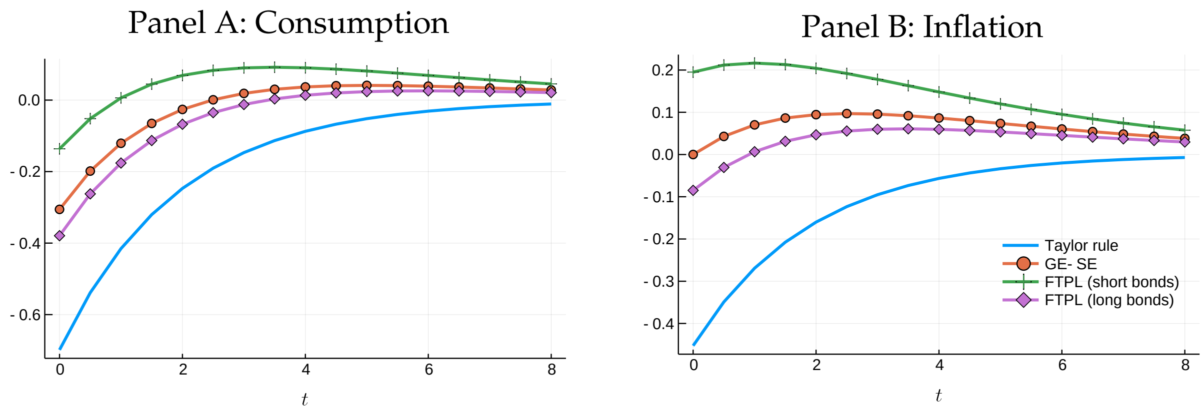

FIGURE 2: Consumption and inflation under different values for the wealth effect

Calibration: $\sigma = 1$, $\kappa = 0.09$, $\rho = 0.01$, and $\tau = 0.25$. Public debt is set to 100% of GDP and the duration of long-term bonds is set to five years. Half-life of nominal interest rate is four months.

that the average level of consumption does not change. Another important distinction is that the consumption under the GE-SE has both backward-looking and forward-looking components. Therefore, the economy does not go immediately to steady state if the nominal interest rate returns to its steady-state level. The differences between the Taylor rule equilibrium and GE-SE become even starker when we consider the behavior of inflation, to which we turn next.

2.3 The Determinants of Inflation

Proposition 1 presents the decomposition of consumption into a substitution effect, a wealth effect, and an interaction term. There is a similar decomposition of inflation.

Proposition 2 (Inflation Decomposition)**.** *In the bounded solutions to the system* (1)-(2)*, inflation is given by*

$$\pi_t = \pi_t^S + \frac{\kappa}{\rho} e^{\omega t} C, \tag{12}$$

where $\{\pi_t^S\}_{t=0}^{\infty}$ *is the solution to* (10)-(11)*, and* C *is the average consumption as defined in* (8)*.*

In $t = 0$*,*

$$\pi_0 = \frac{\kappa}{\rho} C.$$

The decomposition uncovers an important result: inflation in period 0 is completely determined by the wealth effect rather than by the substitution effect. That is, inflation on impact does not depend on the change in initial consumption, but on whether the households' lifetime consumption is on average higher or lower after the shock. Initial inflation depends on whether households are richer or poorer rather than on the timing of the consumption path.

To understand this result, it is important to note the forward-looking nature of the New Keynesian Phillips Curve, which depends only on the net present value of *future* consumption for the determination of inflation today. Since the present value of the Hicksian demand is zero, initial inflation is determined solely by the wealth effect. In particular, the old-Keynesian view that lowering consumption in a period is enough to lower inflation in that period does not apply to New Keynesian environments.

Hence, we have that $\pi_0^S = 0$ regardless of the path or magnitude of the nominal interest rates. In the absence of wealth effects, the monetary authority is unable to control initial inflation. Moreover, inflation has *Neo-Fisherian* features under the GE-SE, as an increase in nominal interest rates actually *raises* inflation,[21]

$$\frac{\partial \pi_t^S}{\partial i_s} > 0,$$

for $t > 0$. Therefore, the inverse relation between the nominal interest rates and inflation under the Taylor rule equilibrium is driven entirely by negative wealth effects. In the absence of such wealth effects, not only does the monetary authority lose control of initial inflation, but the sign of the effect is the opposite of the standard Taylor rule result, as indicated in Figure 2 (Panel B).

Lastly, our decomposition also uncovers the forces determining the long-run dynamics of the model. An often emphasized property of the Taylor rule equilibrium is its long-run monetary neutrality, that is, the result that if nominal interest rates revert back to steady state, then consumption and inflation return to steady state as well. The next lemma show that, actually, *all* bounded solutions of the New Keynesian system share this property.

Lemma 3 (Long-run monetary neutrality)**.** *Suppose* $\lim_{t\to\infty} i_t = \rho$*. Then, all bounded solutions*

[21] See Appendix B for a formal derivation of this result.

of the system (1)-(2) *satisfy*

$$\lim_{t\to\infty} \pi_t = \lim_{t\to\infty} c_t = 0.$$

An implication of Lemma 3 is that the long-run properties of consumption or inflation cannot be used to select an equilibrium, as all equilibria share the same long-run behavior.

2.4 The Intertemporal Keynesian Cross

The previous analysis emphasizes the important role that the wealth effect plays in the equilibrium of the economy. Here, we study the determination of wealth effects. In particular, we explore whether we can tie the determination of C to observables, without resorting to policy rules.

There are two main forces that determine the equilibrium average consumption: the *spending-income spiral* and the *spending-inflation spiral*, given, respectively, by

$$\rho \int_0^\infty e^{-\rho t} y_t dt = C, \qquad \pi_t = \pi_t^S + \frac{\kappa}{\rho} e^{\underline{\omega} t} C.$$

The spending-income spiral states that average income equals average consumption, and higher income leads to higher consumption. The spending-inflation spiral states that, given a path for the nominal interest rate, the inflation rate increases with average consumption. Plugging in these two relations into (8), we get

$$C = [1 - (\tau - \sigma \underline{\omega} b)]C + \rho \int_0^\infty e^{-\rho t}[b(i_t - \pi_t^S - \rho) + T_t]dt. \tag{13}$$

Equation (13) states that average consumption is determined according to an *Intertemporal Keynesian Cross*, in the spirit of the old-Keynesian logic found in many introductory textbooks.[22] In fact, one could interpret $1 - (\tau - \sigma \underline{\omega} b)$ as analogous to the marginal propensity to consume (MPC), and $\rho \int_0^\infty e^{-\rho t}[b(i_t - \pi_t^S - \rho) + T_t]dt$ as the autonomous portion of spending. To simplify the notation, define

$$W^F \equiv \int_0^\infty e^{-\rho t} b(i_t - \pi_t^S - \rho)dt, \qquad T \equiv \int_0^\infty e^{-\rho t} T_t dt,$$

[22] Note that our definition of an Intertemporal Keynesian Cross is different from the one in Auclert et al. (2018) or the New Keynesian Cross in Bilbiie (2019).

and $A \equiv W^F + T$, where A is the autonomous spending, W^F is the *Hicksian financial wealth* (since it is calculated using the inflation rate from the substitution effect), and T is the present value of transfers.

To determine the equilibrium value of C, we need to consider two separate cases: i) monetary policy has no fiscal consequences, that is, $\tau = b = 0$; ii) monetary policy has fiscal consequences, that is, either $\tau > 0$ or $b > 0$ (or both). The equilibrium implications of the model are very different in these two cases.

Consider first the case $\tau = b = 0$. This is a knife-edge case and not empirically relevant, but it still important to consider it, as it is commonly assumed in the literature. Evaluating equation (13) at $\tau = b = 0$, we get

$$C = C + \rho A \Rightarrow A = T = 0,$$

that is, the only restriction we get from this equation is that the present value of transfers must be zero. But beyond that, the budget constraint of the household imposes no restriction on what average consumption is. In particular, the level of average consumption, and hence the wealth effect, has a self-fulfilling nature: if agents expect to receive higher income, $\int_0^\infty e^{-\rho t} y_t dt$, they increase their consumption accordingly, and since output is demand determined, output increases to satisfy that demand. But since the households' income equals the present value of output, the increase in consumption becomes self-fulfilling. This logic resembles the case in which the MPC is equal to one in old-Keynesian analysis. In the standard equilibrium selection, the Taylor rule pins down C by imposing that only a specific path of inflation be consistent with a bounded equilibrium. However, this result presents a challenge to testing the theory since, given observables, a continuum of paths for consumption and inflation (indexed by C) is consistent with the system of equations governing the equilibrium.

However, the indeterminacy of the wealth effect disappears when monetary policy has fiscal consequences, and fiscal data can be used to discipline the model. As we move away from $\tau = b = 0$, average consumption is determined by the *observed paths* of policy variables. Suppose $\tau > 0$ or $b > 0$. The next proposition shows how to determine the average consumption.

Proposition 3 (Intertemporal Keynesian Cross)**.** *Suppose $\tau > 0$ or $b > 0$ (or both). Average consumption, C, is given by*

$$C = \rho \frac{A}{\tau - \sigma \underline{\omega} b} = \rho \frac{W^F + T}{\tau - \sigma \underline{\omega} b}.$$

To grasp the intuition behind this result, consider the impact of a shock that increases the value of autonomous spending by $\frac{\Delta}{\rho}$. If we were to keep inflation and output constant, this would generate an increase of consumption of Δ. But higher consumption raises demand, increases the households' income by $1 - \tau$, and generates inflation, reducing the real return on the household's assets by $\sigma \underline{\omega} b$ (remember $\underline{\omega} < 0$). As a result, there is a (first-round) net increase in wealth of $1 - (\tau - \sigma \underline{\omega} b)$. This additional wealth further increases consumption, which increases net wealth again, in the following way

$$\Delta + (1 - (\tau - \sigma \underline{\omega} b))\, \Delta + (1 - (\tau - \sigma \underline{\omega} b))^2\, \Delta + \ldots = \frac{\Delta}{\tau - \sigma \underline{\omega} b}$$

Thus, an intuition analogous to the standard old-Keynesian cross is useful for thinking about wealth effects in the New Keynesian model.

Note that Proposition 3 states that, given a path for the nominal interest rate and government transfers, average consumption is determined by equation (13). This is an important result for two reasons. First, it shows that, given the *equilibrium* path for fiscal variables, the New Keynesian model has a unique prediction for the paths of consumption and inflation. This result is in stark contrast to the multiplicity of equilibria obtained when fiscal variables are treated as a residual.[23] Thus, it is important to incorporate the fiscal side of the model even if one believes that it is of the passive or "Ricardian" type. Second, it shows that, in this benchmark RANK model, the wealth effect is completely determined by fiscal variables. The model has no other channel through which a monetary shock can affect households' wealth, as it operates either through government bonds or government transfers.

The idea that the liabilities of the government are the relevant assets for the assessement of wealth effects is not new. This observation was central to Pigou's argument in his response to Keynesian economics. For instance, Patinkin describes Pigou's argument as

[23]It is important to note that this result is not an implication of the FTPL. All these results are consistent with any monetary/fiscal regime that generates the given equilibrium paths for policy variables.

follows:[24]

> (...) the private sector considered in isolation is, on balance, neither debtor nor creditor, when in its relationship to the government, it *must be* a net "creditor."(...) If we assume that government activity is not affected by the movements of the price level, then the net effect of a price decline must always be stimulatory.

Two aspects of this quote are important. First, the idea that private assets cancel out in the aggregate, but households are on net creditors of the government. Second, the fact that it is assumed that "government activity is not affected" by the shock. The Pigou effect, as described here, is remarkably similar to the modern formulation of the FTPL. In both cases, the assumption that fiscal variables do not react is important, as is the assumption of flexible prices, as the effect comes from movements in the price level. In contrast to both the original Pigou effect and the FTPL, our focus here is on the dynamics under sticky prices. The wealth effect will not come, then, from adjustments in the price level, but from movements in the real interest rate, as we discuss next.

Wealth effects in the FTPL. In the spirit of the original formulation of the FTPL, and consistent with Pigou's analysis described above, we assume that there is no reaction of the fiscal authority to changes in the nominal interest rate, i.e., $T_t = 0$ for all $t \geq 0$. In contrast to both, we assume prices are sticky. An important implication of these assumptions is that an increase in interest rates generates a *positive* wealth effect, at least in the case of short-term bonds we have considered so far.

Lemma 4 (FTPL-Pigou effect)**.** *Suppose $\tau > 0$ or $b > 0$ (or both), and $T = 0$. Suppose government debt pays coupons e^{-mt} at period t. Then, there exists a threshold $m^* > 0$ such that*

$$\frac{\partial C}{\partial i_t} < 0 \tag{14}$$

if and only if the maturity of government debt is sufficiently long, i.e. $m < m^*$.

[24] See Patinkin (1948).

Lemma 4 considers the wealth effect under the FTPL. To highlight the importance of the maturity of the government debt, we extend the basic model to allow for exponentially decaying coupons for government bonds, as in Woodford (2001). The rate of decay m is inversely related to the maturity of bonds. A perpetuity corresponds to $m = 0$, while $m \to \infty$ corresponds to the short-term bonds we have assumed until now. Introducing long-term bonds brings a new effect, as an increase in the nominal interest rates reduces the value of bonds when they have positive duration, with important implications for the determination of wealth effects.

Consider first the case in which the maturity of government bonds is relatively short. The wealth effect in response to an increase in the interest rate is positive in this case. As an increase in nominal rates leads to an increase in real rates when prices are sticky, households reinvest their savings at higher real rates after the shock. This positive wealth effect explains the consumption boom after two quarters in response to an increase in nominal interest rates observed in Panel A of Figure 2, as the positive wealth effect eventually overturns the substitution effect.[25] The Neo-Fisherian response is even more pronounced than in the GE-SE equilibrium, generating a sharp increase in inflation, as seen in Panel B of Figure 2.

The previous result relies on the assumption of the short maturity of government bonds. An increase in nominal interest rates reduces the value of government bonds when they are long term, generating a reduction in wealth for households. If this effect is strong enough, which depends on the duration of the public debt, then an increase in interest rates generates a *negative* wealth effect. Consumption and inflation drop on impact in response to the increase in nominal interest rates in our calibrated example. The negative wealth effects generated by government bonds overturn the Neo-Fisherian predictions, but the effects are weaker than those generated under a Taylor rule.

In a recent paper, Cochrane (2018) considers the implications of an FTPL model with sticky prices and long-term bonds. He also finds that inflation falls in response to an increase in nominal interest rates. He argues that the logic behind the result is, however, substantially different from the standard New Keynesian logic. By focusing on the role of

[25] In the case of $\tau = 0$, such that the primary surplus is exogenous as is usually assumed in the context of the FTPL, consumption increases at all dates in response to higher interest rates.

wealth effects, we show that the two cases are more similar than they seem to be at first. The FTPL with long-term bonds relies on financial wealth, W^F, to generate a negative wealth effect. The negative wealth effect in the Taylor rule relies, generically, on movements in T. In the empirically relevant case of $\tau > 0$ or $b > 0$, a fiscal response is required to achieve the negative wealth effects in the Taylor rule equilibrium. Fiscal variables play an important role in achieving a negative wealth effect in both cases, but one theory emphasizes the role of government bonds, while the other depends on movements in fiscal transfers.

2.5 An alternative consumption decomposition

An alternative decomposition to ours divides the response of equilibrium consumption into a *direct effect* of the real interest rate, keeping output and fiscal policy fixed, and a *indirect effect* that incorporates the changes in output and fiscal policy:

$$c_t = \underbrace{c_t^H + \rho \int_0^\infty e^{-\rho t} b(i_t - \rho - \pi_t) dt}_{\text{direct effect}} + \underbrace{\rho \int_0^\infty e^{-\rho t} \left[(1-\tau) y_t + T_t \right] dt}_{\text{indirect effect}} .$$

This decomposition appears in, for example, Kaplan et al. (2018). There are two main differences between this decomposition and the one proposed in Proposition 1. First, the direct effect captures the response of the *Marshallian demand* to the equilibrium real interest. Compared with (9), the direct effect combines the Hicksian demand with the wealth effect coming from interest payments. Second, we further decompose the Hicksian demand into a GE-SE and a GE amplification. This allows us to separate the interactions between the wealth effect and substitution effect in general equilibrium. Therefore, the two decompositions will coincide only if $b = 0$, so there is no distinction between the Marshallian and the Hicksian demand, and prices are rigid, so the GE multiplier is equal to zero.

These differences reflect the different goals that the decompositions were designed to achieve. The direct effect isolates the impact of changes in *real* interest rates, both from an intertemporal substitution channel and from revaluation effects from asset holdings. Proposition 1 focuses on the change in the *nominal* interest rate and the impact it has on the *general equilibrium* of the economy. A key innovation of our decomposition is that our substitution effect incorporates only the change in prices that is consistent with the substitution effect. It is this feature that allows us to identify the importance of the wealth effect in the equilibrium dynamics of the economy.

3 The Quantitative Importance of Wealth Effects in RANK

In this section, we study the quantitative importance of wealth effects in a medium-scale DSGE New Keynesian model. We first extend our consumption decomposition to the DSGE model and show that wealth effects play an important role in explaining consumption dynamics. We then provide empirical estimates of the two main drivers of wealth effects in a RANK model: *i*) revaluation of government bonds, and *ii*) fiscal transfers. We find that our empirical estimates differ significantly from the corresponding fiscal behavior in the DSGE model, suggesting that the fiscal-based mechanism implicit in the RANK model is not strong enough to generate the empirically required wealth effects.

3.1 The Model

The model is a cashless limit variant of Christiano et al. (2005) augmented to explicitly account for fiscal variables. Time is discrete and denoted by $t = 0, 1, 2, \ldots, \infty$. The economy is populated by a continuum of mass one of infinitely-lived households. Households derive utility from the consumption of a final good and leisure. Their preference for consumption exhibits an external habit variable. Labor supply is differentiated across households. Wages for each type of labor are negotiated by a union, which chooses the wage but is subject to nominal rigidities à la Calvo. Households are the owners of the capital of the economy. They rent capital services to the firms, which are a function of the capital stock they hold and the utilization level they choose. A higher utilization level comes at the cost of higher depreciation. Households also decide how much capital to accumulate given the adjustment costs they face.

There are two types of firms in the economy. There is a continuum of intermediate goods producers, which transform labor and capital services into differentiated goods and set prices subject to the Calvo friction. Those wages and prices that cannot be re-optimized in a given period are indexed to past inflation. Intermediate goods producers are subject to a sales tax. The second type of firm is a representative firm that produces the final consumption good using the intermediate goods as inputs and sells the output in competitive markets. Finally, there is a government that chooses a path for the nominal interest rate, sales taxes, lump-sum transfers, and debt. Importantly, unless otherwise noted, we assume that monetary policy follows a standard Taylor rule. The reader can refer to the appendix for a detailed derivation of the model.

To determine the necessary fiscal response to a monetary shock, we explicitly introduce the government's budget constraint. A log-linear approximation of the budget constraint around a steady state with constant inflation and positive government debt is given by

$$b_y b_t = \frac{1+i}{(1+\pi)(1+g)} b_y (i_{t-1} - \pi_t - \rho + b_{t-1}) - (\tau y_t + \tau_t - T_t) \tag{15}$$

where y_t is output, i_t is the nominal interest rate (set by the monetary authority in period t), π_t is the inflation rate, b_t is the real value of government bonds outstanding in period t, τ_t is a sales tax, and T_t is a lump-sum transfer. Variables without a subscript denote the corresponding variables in steady state, g denotes the growth of the economy in a balanced growth path, and ρ denotes the households' subjective discount rate. Finally, b_y denotes the debt-to-GDP ratio in steady state. Note that, given that monetary policy is specified using a Taylor rule, the only role of the government's budget constraint (15) is to determine the present value of fiscal transfers $\{T\}_{t=0}^{\infty}$. Although it is not necessary to compute the solution of the model, this model-implied fiscal response is relevant because it can be directly compared to the corresponding fiscal response in the data, providing a way to empirically assess the ability of the model to generate wealth effects.

3.2 Empirical Evidence on the Fiscal Response to Monetary Shocks

Following Christiano et al. (2005) and Altig et al. (2011), we estimate the parameters of the model using *impulse-response matching*, that is, the parameters are chosen to minimize the distance between the model impulse responses to monetary policy shocks and the corresponding impulse responses observed in the data. We estimate the empirical IRFs using a VAR identified by a recursiveness assumption, as in Christiano et al. (1999), extended to include fiscal variables.

VAR estimation. The variables included in the VAR are: real GDP per capita, CPI inflation, real consumption per capita, real investment per capita, capacity utilization, hours worked per capita, real wages, tax revenues over GDP, government expenditures per capita, federal funds rate, and the real value of government debt per capita. Following Christiano et al. (2005), we estimate a four-lag VAR using quarterly data for the period 1965:3-1995:3. The identification assumption of the monetary shock is as follows: the only variables that are allowed to react contemporaneously to the monetary policy shock are the federal funds rate and the value of government debt. All other variables, including government tax revenues and expenditures, are allowed to react with a lag of one quarter. This assumption is the natural extension of Christiano et al. (1999): while agents' decisions (with agents, in our case, including households and the government) cannot react to the shock contemporaneously, financial variables (in our case, the federal funds rate and the value of government debt) immediately incorporate the information of the shock.

All the variables are obtained from standard sources (see the appendix for the details), except for the real value of debt, which we construct from the series provided by Hall et al. (2018).[26] These data provides the market value of government debt held by private investors at a monthly frequency from 1776 to 2018. We transform the series into quarterly frequency by keeping the market value of debt in the first month of the quarter. This choice is meant to avoid capturing changes in the market value of debt arising from changes in the *quantity* of debt after a monetary shock instead of changes in *prices*.

Figure 3 shows the results. As is standard in the literature, we find that a contractionary monetary shock increases the federal funds rate and reduces output and inflation on impact.[27] Moreover, the contractionary monetary shock reduces consumption, investment, and hours worked.[28] The bottom three graphs show the dynamics of fiscal variables. We find that a monetary shock has a statistically significant negative effect on revenues and

[26]For recent work using a similar data construction, see e.g., Cochrane (2019) and Jiang et al. (2019).

[27]As is common in the literature, our point estimates reveal a price puzzle on impact. However, this result is not statistically different from zero at a 95% confidence level.

[28]The IRFs imply a statistically significant expansion after 10 quarters, but this result varies across specifications. In a VAR without the fiscal variables, for instance, we do not find a statistically significant expansion after 10 quarters.

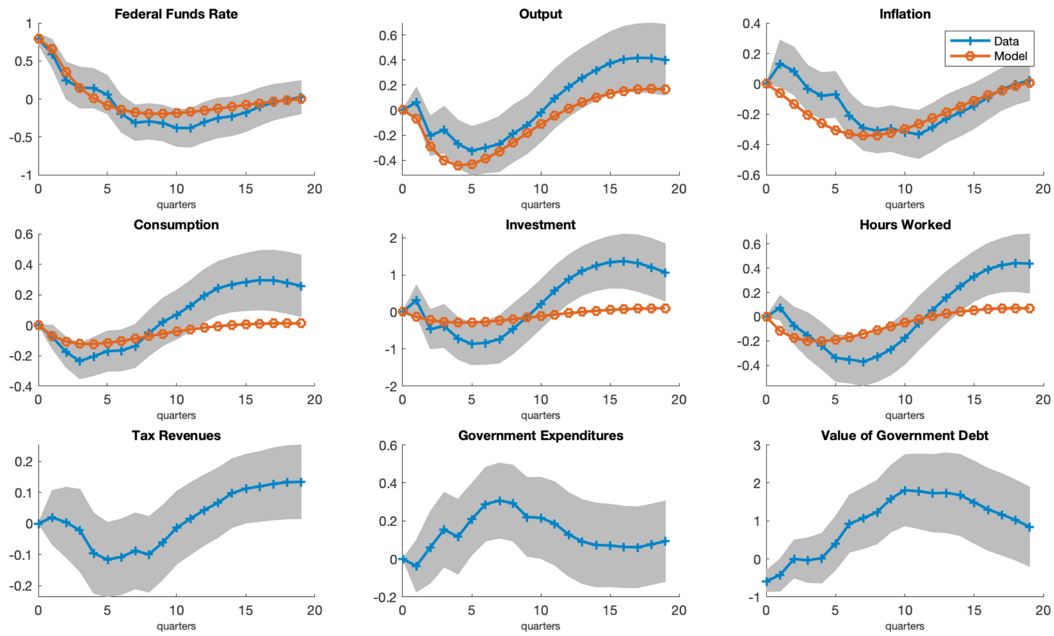

FIGURE 3: Impulse responses to a monetary shock. Gray area represents 95% confidence intervals.

a positive effect on expenditures. This effect is likely driven by the automatic stabilizer mechanisms embedded in the government accounts. Since the monetary shock is contractionary, households' income and employment decrease. This has three effects. First, government revenues decrease. Second, because income taxes are progressive, the average income tax in the economy decreases. Third, because a large fraction of government transfers are unemployment benefits and other safety net transfers, expenditures increase during recessions. All these channels generate a reduction in the government's primary surplus. The graph showing government debt shows how the reduction in the primary surplus is financed. A contractionary monetary shock generates a large reduction in the market value of government debt: on impact, initial debt declines by 44 basis points (bps). Immediately after, the government accumulates debt for about 12 quarters.

Model impulse responses. We use the empirical IRFs to estimate the parameters of our medium-scale DSGE model using "impulse-response matching" techniques.[29] Our estima-

[29] See Christiano et al. (2005) for a detailed description and Christiano et al. (2010) for a Bayesian version of the estimator.

tion generates parameter values consistent with those found in the literature. We relegate to the appendix a detailed description. Figure 3 shows the model-based impulse responses to a monetary shock. The results are roughly in line with those obtained by Altig et al. (2011). The model does a good job of matching the dynamics of the federal funds rate, output, and inflation, but it faces more difficulty in capturing the dynamics of consumption, investment, and hours, especially after 10 quarters.

3.3 Consumption decomposition in the DSGE model

Our first exercise is to decompose the equilibrium dynamics of consumption into a substitution effect, a wealth effect, and an interaction term. As in Section 2, we obtain this decomposition by dropping the Taylor rule and considering how consumption is directly related to the nominal interest rate for a given level of average consumption. The result of this decomposition is provided in the next proposition.

Proposition 4 (Consumption Decomposition in General Equilibrium (DSGE)). *Suppose a generalized Blanchard-Kahn condition, defined in the appendix, is satisfied.*[30] *Then,*

$$c_t = \underbrace{c_t^S}_{\text{GE-SE}} + \underbrace{C}_{\text{wealth effect}} + \underbrace{\overbrace{\left(\sum_{k=1}^{p+1} \nu_k \lambda_k^t - 1\right)}^{\text{GE multiplier}} \times C}_{\text{GE amplification}},$$

where c_t^S is a function of $\{i_t\}_{t=0}^{\infty}$ and it is independent of C, p is the number of predetermined variables in the system, and $\lambda_k < 1$, for $k = 1, \ldots, p+1$.

Proposition 4 extends the result in Proposition 1 to the context of the richer DSGE model. It decomposes the equilibrium consumption into the same three components: the *GE-SE*, the *wealth effect*, and the *GE amplification*. The GE-SE has properties analogous to those derived in the context of the simple New Keynesian model of Section 2. First, the present value of GE-SE is equal to zero when evaluated using steady-state prices. Second, as shown in the appendix, it corresponds to the solution to a fixed-point problem, as

[30] Intuitively, this condition says that the system lacks exactly one boundary condition without the Taylor rule. Therefore, the indeterminacy can be indexed by the wealth effect C.

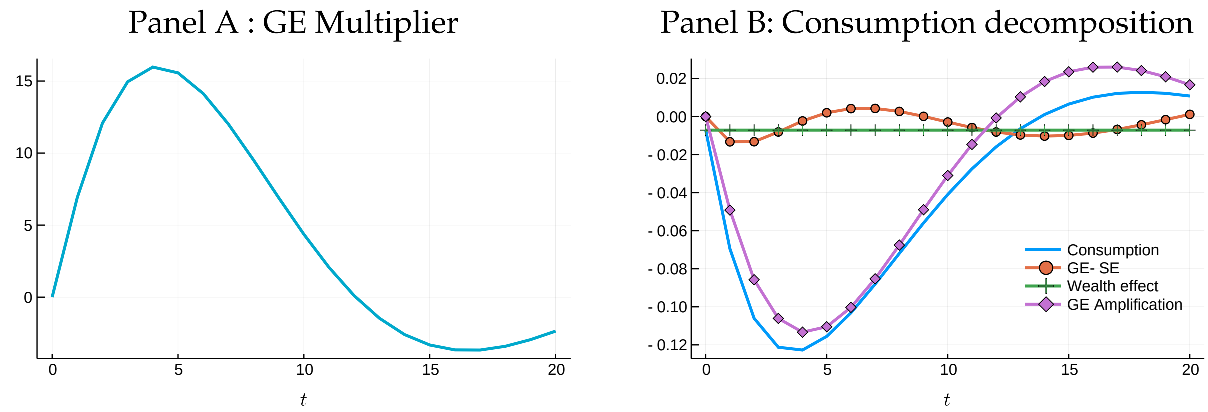

FIGURE 4: GE Multiplier and consumption decomposition in the DSGE model

it equals the Hicksian demand evaluated at the inflation rate that is itself consistent with the Hicksian demand. Moreover, the GE-SE and the GE multiplier are independent of the wealth effect C, so the GE-SE is uniquely determined by the sequence of nominal interest rates. The GE multiplier in the DSGE model has richer dynamics than the multiplier obtained in the simple model. In particular, it can have hump-shaped dynamics, an important feature for matching the sluggish response of consumption observed in the data.

A simple way of computing the decomposition is the following. Given a path of nominal interest rates, the GE-SE can be computed by replacing the Taylor rule by the condition that average consumption is equal to zero.[31] Given c_t and c_t^S, the GE Multiplier can be obtained as follows:

$$GE\ Multiplier_t = \frac{c_t - c_t^S}{C} - 1.$$

The GE amplification term is then $GE\ Multiplier_t \times C$.

Figure 4 depicts the results. The GE-SE and the wealth effect have a marginal role in the equilibrium dynamics of the economy. While equilibrium consumption has a peak response of 12 bps, the GE-SE generates a peak response slightly above 1 bp, while the wealth effect is constant at less than 1 bp.[32] That is, the GE-SE and WE jointly account for less than 17% of the total response of consumption at their peak. It is the interaction term that is driving most of the dynamics. While the wealth effect is small, the general

[31] In Dynare, for instance, this could be easily implemented by introducing, in place of the Taylor rule, the equation $c_t^S + b_t = \frac{1}{\beta} b_{t-1}$, given $b_{-1} = 0$, which represents the condition $C = 0$ in recursive form.

[32] The peak response of the GE-SE and overall consumption do not occur in the same period.

equilibrium mechanisms in the model greatly amplify this effect, with a maximum value of the multiplier of 16. That is, at the peak response of consumption, the interaction term contributes around 11 of the 12 bps. These results reinforce our previous analysis: the economy reacts little to the change in the path of the nominal interest rate but substantially to the resulting change in the households' wealth.

An implication of the results in Figure 4 is that the quantitative performance of the model relies on generating sufficient wealth effects. In particular, the response of consumption is substantially dampened in the absence of wealth effects, as can be seen from the GE-SE. As discussed in Section 2.4, wealth effects in RANK need to be supported by an appropriate fiscal response. We can then assess the ability of the model to generate wealth effects by comparing the fiscal response required by the model and the corresponding fiscal response estimated in the data.

3.4 Assessing the Quantitative Importance of Wealth Effects

We next present the main exercise of this section. We assess whether the fiscal response to a monetary shock estimated in the data is enough to support the wealth effects required by the model. We do this in two steps. First, we test whether the empirical fiscal response and the one implied by the model are statistically different from each other. Second, we consider the quantitative impact on the model impulse responses of having to exactly match the fiscal response observed in the data.

Empirical and model-based fiscal responses. The fiscal response in the model corresponds to the present discounted value of fiscal transfers over an infinite horizon, that is, $\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t T_t$. We next consider the empirical counterpart of this quantity. First, rewrite the per-period budget constraint (15) as an intertemporal budget constraint from period zero to $\mathcal{T}$:

$$\underbrace{b_y b_0}_{\substack{\text{debt}\\ \text{revaluation}}} = \sum_{t=0}^{\mathcal{T}}\left(\frac{1}{1+\rho}\right)^t\left[\underbrace{\tau y_t+\tau_t}_{\text{tax revenue}} - \underbrace{\frac{b_y}{1+\rho}(i_{t-1}-\pi_t-\rho)}_{\text{interest payments}}\right] - \underbrace{T_{0,\mathcal{T}}+\left(\frac{1}{1+\rho}\right)^{\mathcal{T}} b_y b_{\mathcal{T}}}_{\substack{\text{other transfers/expenditures}\\ \text{\& final debt}}} \quad (16)$$

where $\frac{1}{1+\rho} = \frac{1+i}{(1+\pi)(1+g)}$. The right-hand side of (16) is the present value of the impact of a monetary shock on the fiscal accounts. The first term represents the change in revenues due to the real effects of monetary shocks. If a contractionary monetary shock generates a recession, government revenues will naturally decrease as a consequence, both because output decreases and because the average tax decreases if the tax system is progressive. The second term represents the change in interest payments on government debt due to the change in nominal rates. For example, a contractionary monetary shock increases the nominal payments on government debt. The last two terms are adjustments in transfers and other government expenditures, and the final debt position at period $\mathcal{T}$, respectively. In particular, $T_{0,\mathcal{T}}$ represents the present discounted value of transfers from period 0 through $\mathcal{T}$. Provided that $\mathcal{T}$ is large enough, such that (y_t, τ_t, i_t) have essentially converged to the steady state, then the value of debt at the terminal date, $b_{\mathcal{T}}$, equals (minus) the present discounted value of transfers and other expenditures from period $\mathcal{T}$ onward. Hence, the last two terms combined can be interpreted as the present discounted value of fiscal transfers from zero to infinity.

The left-hand side represents the revaluation effect of the *initial* stock of government debt. In the presence of long-term bonds, a contractionary monetary shock reduces the initial value of government bonds. Hence, part of the adjustment in response to the shock comes from a reduction in the value of debt, instead of coming entirely from raising present or future taxes. We define the *fiscal needs of the government* as the sum of the present value of interest payments minus the tax revenue, which equals (minus) the sum of the present value of transfers and the initial value of debt.

In a model with a Taylor rule, the government's fiscal needs are independent of the maturity of government debt, as output and inflation are independent of the government's budget constraint. Hence, without loss of generality, we compute the government's fiscal needs in the model by assuming that bonds are short-term. Different assumptions about government maturity affect the breakdown between transfers and initial debt, but not the total fiscal needs of the government.

Table 1 shows the impact on the fiscal accounts of a monetary policy shock, both in the data and in the estimated model. We start by testing whether our estimate of the fiscal

	(1) Revenues	(2) Interest Payments	(4) - (3) - (5) Fiscal Needs	(3) Transfers & Expenditures	(4) Debt in T	(5) Initial Debt	(1) - (2) - (3) + (4) - (5) Residual
Data	35.97	25.19	15.71	41.70	13.63	-43.77	-26.48
	[-55.49,67.56]	[-22.40,72.78]	[-28.12,59.53]	[-13.38,96.78]	[-36.11,63.37]	[-63.22,-24.33]	[-128.52,75.55]
Model	-2.40	70.09	72.90	-	-	-	-

TABLE 1: The impact on fiscal variables of a monetary policy shock

Note: Confidence interval at 95% confidence level.

response to a monetary shock is consistent with the government's intertemporal budget constraint. To test this, we apply equation (16) to the data and check whether the difference between the left-hand side and the right-hand side is different from zero. We decompose the fiscal response in the data into six groups: present value (PV) of revenues, PV of interest payments, PV of transfers and expenditures, final value of debt, initial value of debt, and a residual. The residual is calculated as

$$\text{Residual} = \text{Revenues - Interest Payments - Transfers + Debt in } \mathcal{T} \text{ - Initial Debt}$$

We truncate the calculations to quarter 60, that is, $\mathcal{T} = 60$ in equation (16). The results in Table 1 imply that we cannot reject the possibility that the residual is zero and, therefore, we cannot reject the possibility that the intertemporal budget constraint of the government is satisfied in our estimation.

The adjustment of the fiscal accounts in the data corresponds to the patterns we observed in Figure 3. The contractionary monetary policy shock leads to an increase in the present value of interest payments and of transfers and expenditures. The present value of revenues goes up in response to the shock, due to the boom generated by the monetary shock after period 10. The response of initial debt is quantitatively important, and it accounts for the bulk of the adjustment in the fiscal accounts.

The second line of Table 1 shows the corresponding fiscal variables that come out of our quantitative model. Since the model forces the intertemporal budget constraint to be satisfied by construction, and we collapse debt and transfers into a unique variable, only the first three columns are relevant. The values of revenues and interest payments are both within the 95% confidence interval of the data. The main implication of Table 1 is that the

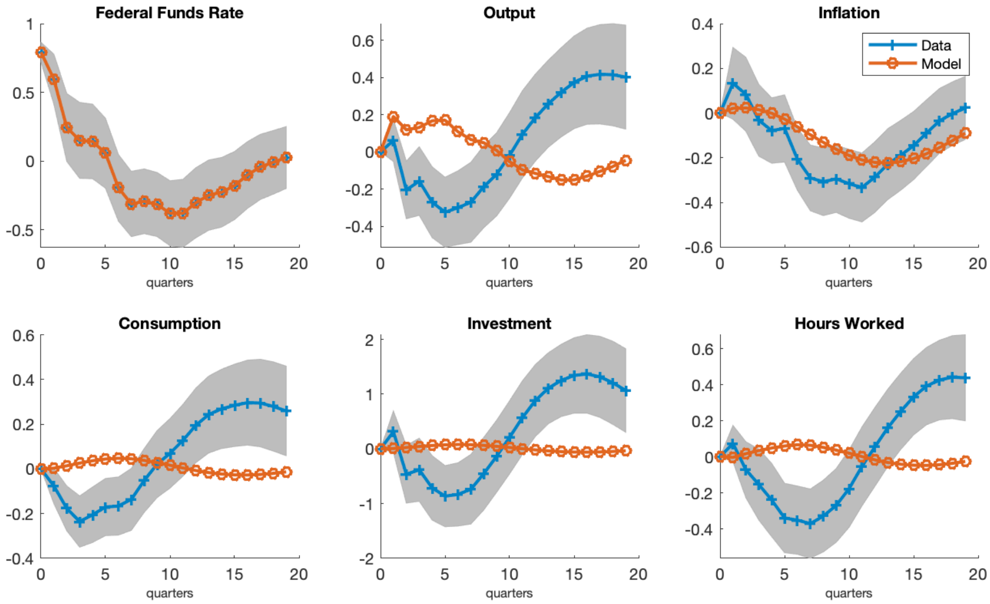

FIGURE 5: Model impulse response functions to a monetary shock. Interest rates and fiscal variables match the data. Gray area represents 95% confidence intervals.

fiscal needs observed in the data are substantially smaller, and statistically different, from the one required by the model. While the government's fiscal needs are roughly 73 bps in the model, in the data this number is below 16 bps. Therefore, the empirical fiscal response is substantially weaker than the response required by the model.

Even though the result in Table 1 implies that the model-based government's fiscal needs are statistically different from those in the data, this evidence is not enough to conclude that this difference is economically significant. On the one hand, given that the marginal propensity to consume is relatively low in the standard calibration, it is possible that this difference generates only small differences in consumption. On the other hand, movements in fiscal variables, and ultimately wealth effects, are substantially amplified in general equilibrium. To evaluate the economic significance of the discrepancy in fiscal needs we documented in Table 1, we next consider the impact of these differences in the model impulse responses.

Fiscal needs and model impulse responses. In Figure 3, we considered the model impulse responses when the average consumption is determined by a Taylor rule. As we have seen, this level of consumption requires a counterfactual fiscal response to be supported in equilibrium. We next consider the model impulse responses when the government's fiscal needs are consistent with the response we estimated in the data. Figure 5 depicts the results. The solid line is the impulse response estimated from the data, with the gray area being the 95% confidence intervals. The pointed line is the impulse response of the model. We can see that the model predicts higher inflation than does the data, though mostly inside the confidence bands. However, the data clearly rejects the impulse response for output. While the data implies that a positive monetary shock generates a recession in the short run, the model predicts a boom. The explanation for the initial boom is reminiscent of our analysis of the FTPL in Section 2.4. In the absence of a fiscal response, an increase in nominal interest rates raises consumption and inflation due to a positive wealth effect, as the increase in real rates raises the households' interest income. To avoid this outcome, the model requires a strong enough reduction in fiscal transfers or a sufficient decline in the initial value of government debt. Our empirical results indicate that the movements in fiscal transfers or initial debt are not strong enough in the data to overcome the positive wealth effect, at least in the context of a RANK model.[33]

Thus, we derive two main conclusions from this analysis. First, wealth effects play an important role in the ability of the DSGE model to match the observed response to monetary policy shocks, as shown in Figure 4. Second, the model relies on a counterfactual fiscal response to support these wealth effects, as shown in Figure 5. The response of fiscal variables to monetary shocks observed in the data is too weak compared with what is required by the model.

Our results in this section focus on the role of *fiscal-based* wealth effects, as these are generically the only kind of wealth effects in RANK models. As we pointed out in Section 2.4, such fiscal-based wealth effects have a long tradition in monetary economics, going back to the work of Pigou. In the same way, skepticism about the empirical importance of this channel is not new. James Tobin argued that wealth effects generated by private assets (or *inside assets*), as emphasized in the work of Irving Fisher, could be much stronger than those generated by the net assets available to households, that is, government bonds:[34]

[33]The results in Figure 5 are based on a model estimated without taking into account the fiscal impulse responses. In Appendix D, we re-estimate the model using the fiscal data and obtain similar results.

[34]See Tobin (1982).

> The gross amount of these "inside" assets was and is orders of magnitude larger than the net amount of the base. Aggregation would not matter if we could be sure that the marginal propensities to spend from wealth were the same for creditors and debtors. But if the spending propensity were systematically greater for debtors, even by a small amount, the Pigou effect would be swamped by this Fisher effect.

In the next section, we show that introducing private debt in a model with heterogeneous agents generates stronger wealth effects and significantly improves the quantitative performance of the model, as suggested by Tobin in the quote above.

4 Private Wealth Effects: A Heterogeneous-Agents Model

In this section, we consider the importance of private wealth effects in a heterogeneous-agents economy. We first extend our consumption decomposition to an analytical Two-Agent New Keynesian (TANK) model with positive private liquidity. We then introduce heterogeneous agents and private assets into our medium-scale DSGE model and show that it substantially improves the quantitative performance of the model.

4.1 Consumption decomposition in a TANK model

Model. The economy is populated by two types of households, borrowers and savers. As in the work of Eggertsson and Krugman (2012), borrowers are more impatient than savers and are subject to a borrowing constraint. The aggregate demand block for this model, in log-linear form, consists of an Euler equation for savers

$$\dot{c}_{s,t} = \sigma^{-1}(i_t - \pi_t - \rho),$$

a budget constraint for borrowers

$$c_{b,t} = (1-\alpha)(w_t - p_t + n_{b,t}) + T_{b,t} - (i_t - \pi_t - \rho)d,$$

a market clearing condition for goods

$$\omega c_{b,t} + (1-\omega)c_{s,t} = c_t,$$

and expressions for the real wage, labor supply, and fiscal transfers, provided in the appendix.[35] We define c_t as the aggregate consumption and $(c_{j,t}, n_{j,t}, T_{j,t})$ as, respectively, the consumption, labor supply, and government transfers to household $j \in \{b, s\}$; ρ denotes the discount rate of savers, ω the share of borrowers, $1-\alpha$ the labor share, and d the private debt-to-GDP ratio in steady state.

The Euler equation of savers holds with equality at all periods, as savers are not constrained in equilibrium. In contrast, the borrowing constraint is binding for borrowers, so they simply consume their income, which consists of labor income and fiscal transfers, net of the interest payments on the debt. In particular, borrowers are unable to smooth out movements in borrowing costs, so changes in interest rates have a direct impact on their consumption.

Solving for the labor income of borrowers in equilibrium, and assuming that the transfers they receive are a function of aggregate income, we obtain the following expression for borrowers' consumption:

$$c_{b,t} = \chi_c c_t - \chi_r(i_t - \pi_t - \rho)d, \tag{17}$$

where $\chi_c > 0$ and $\chi_r > 0$ are coefficients defined in the appendix.

The parameter χ_c captures the cyclicality of borrowers' consumption, and it plays a central role in the aggregate consumption dynamics in TANK models.[36] The second term, in contrast, is typically absent from analytical HANK models, which often assume zero private liquidity. As we show in the next proposition, the presence of private assets has important implications for the aggregate dynamics of the economy.

Proposition 5 (Dynamics in the TANK model)**.** *Suppose $\omega\chi_c < 1$. Then, aggregate consumption c_t and inflation π_t satisfy the following conditions:*

[35] Appendix C contains the complete derivation of the model.

[36] See e.g. the discussion in Bilbiie (2018).

1. ***Generalized Euler equation***

$$\dot{c}_t = \tilde{\sigma}^{-1}(i_t - \pi_t - \rho) - \delta c_t + v_t, \tag{18}$$

where $\tilde{\sigma}^{-1}$ is the macro-EIS, $v_t \equiv -\frac{\chi_r d}{1-\omega\chi_c}\left(\dot{i}_t - \rho(i_t - \rho)\right)$ captures the effect of private debt, and $\delta \geq 0$ is the compounding parameter.

2. ***New Keynesian Phillips curve***

$$\dot{\pi}_t = \rho\pi_t - \kappa c_t. \tag{19}$$

3. ***Intertemporal budget constraint***

$$\int_0^\infty e^{-\rho t} c_t dt = \int_0^\infty e^{-\rho t}\left[(1-\tau)y_t + b(i_t - \pi_t - \rho) + T_t\right], \tag{20}$$

where b is public debt, $T_t = \omega T_{b,t} + (1-\omega)T_{s,t}$ denotes total transfers, and $y_t = c_t$.

Proposition 5 shows that aggregate consumption satisfies a *generalized Euler equation*, which differs from the standard Euler equation in three main respects. First, the macro-EIS $\tilde{\sigma}^{-1}$ can differ from the micro-EIS σ^{-1}. In the case $d = 0$, the macro-EIS is larger than the micro-EIS if and only if $\chi_c > 1$, echoing the result in Bilbiie (2019). Second, the Euler equation features *compounding* when $\delta > 0$, which means that current consumption reacts more strongly to future interest rate changes than with the standard Euler equation.[37] In models without private liquidity, compounding is the result of uninsurable income risk combined with countercyclical inequality, that is, $\chi_c > 1$. In contrast, the model with private liquidity generates compounding even with $\chi_c \leq 1$. Third, the presence of private debt adds a new term to the Euler equation, capturing the direct impact of interest rates on the consumption of borrowers. Note that if $d = 0$, then $\delta = 0$ and $v_t = 0$ for all $t \geq 0$. If we also assume that $\chi_c = 1$, then the Euler equation would be identical to the one in

[37]Integrating the Euler equation forward, and supposing $c_T = 0$ at some date $T > t$, we have that $c_t = -\int_t^T e^{\delta(z-t)}\left[\tilde{\sigma}^{-1}(i_z - \pi_z - \rho) + v_z\right] dz$. Hence, current consumption responds more to future consumption when $\delta > 0$.

(1). Therefore, this heterogeneous-agent model differs significantly from the RANK model only in the case of positive private liquidity.

The supply side of the economy, captured by the New Keynesian Phillips curve, is the same as in the RANK model of Section 2. Moreover, the intertemporal budget constraint is also the same as in the representative-agent economy. Because private assets cancel out in the aggregate, the determination of average consumption depends only on the amount of public assets b and fiscal transfers, as in the analysis of Section 2.4. Despite this fact, private assets play an important role in the determination of wealth effects in this economy.

Consumption decomposition and private wealth effects. We turn next to the main result of this section, a consumption decomposition in our heterogeneous-agents economy with positive private liquidity. We start by applying the partial-equilibrium Slutsky decomposition to the consumption of savers, which, combined with the market clearing condition for goods, gives us

$$c_t = (1-\omega)c^H_{s,t} + (1-\omega)C_s + \omega c_{b,t},$$

where $c^H_{s,t}$ and C_s denote the Hicksian demand and the average consumption of savers, respectively.

The first term $(1-\omega)c^H_{s,t}$ denotes the substitution effect. Because borrowers are constrained, they do not respond to compensated changes in interest rates, so the substitution effect is equal to zero for them. Hence, the wealth effect is a combination of the average consumption of savers and the consumption of borrowers. In contrast to the case of RANK, the wealth effect is then time varying with heterogeneous agents. On average, the wealth effect is still equal to average consumption, C, but it typically differs from C at each point in time. As before, the substitution and wealth effects are intertwined in general equilibrium, as variations in C affect inflation and ultimately the substitution effect. The next proposition provides a decomposition of consumption in our TANK model that takes these interactions into account.

Proposition 6 (Consumption decomposition in TANK)**.** *Suppose $\tilde{\sigma} > 0$ and $\chi_c = 1$.*[38] *Let*

[38] As the consequences of having $\chi_c \neq 1$ have been already explored in the TANK literature, we focus on the case $\chi_c = 1$. For completeness, we consider the general case in appendix C.

(c_t^*, π_t^*) denote the solution to (18)-(19) satisfying $C = 0$ and $c_t^{S,*}$ denote the savers' Hicksian demand evaluated at π_t^*. Then, aggregate consumption is given by

$$c_t = \underbrace{(1-\omega)c_t^{S,*}}_{\text{GE-SE}} + \underbrace{\omega c_t^{S,*} - \frac{\omega\chi_r}{1-\omega}\left(i_t - \pi_t^* - \hat{r}^*\right)d}_{\text{private wealth effect}} + \underbrace{C}_{\text{avg. wealth effect}} + \underbrace{\left(\frac{\overline{\omega}_T + \delta}{\rho} e^{\underline{\omega}_T t} - 1\right)C}_{\text{GE amplification}}, \tag{21}$$

where $\overline{\omega}_T > 0$ and $\underline{\omega}_T < 0$ denote the eigenvalues of (18)-(19), and $\hat{r}^* \equiv \rho\int_0^\infty e^{-\rho s}(i_s - \pi_s^*)ds$.[39]

Proposition 6 decomposes consumption into four components: GE-SE, average wealth effect, GE amplification, all of which are also present in the representative-agent economy of Section 2, and the *private wealth effect*, which is present only in an economy with heterogeneous agents. To gain intuition on formula (21), consider first the case $d = C = 0$. Compared to a RANK model, the substitution effect gets weaker, as only a fraction $1-\omega$ of agents engage in intertemporal substitution. As savers shift their consumption over time, the income of borrowers is affected, as output moves with shifts in demand. Borrowers react by changing their consumption, which further affects aggregate income and again borrowers' consumption, in a manner analogous to Samuelson's Keynesian multiplier. In the case $\chi_c = 1$, the private wealth effect is such that the aggregate consumption coincides with the consumption in RANK. As in the "as-if" result of Werning (2015), this illustrates that having heterogeneous marginal propensities to consume does not necessarily alter the aggregate response of the economy to changes in interest rates.

Consider now the case $d > 0$ and $C = 0$. Aggregate consumption differs from the representative-agent benchmark despite our assumption of $\chi_c = 1$. An increase in interest rates generates a negative wealth effect for borrowers and a positive wealth effect for savers. In present value terms, the negative impact on borrowers exactly cancels out the positive impact on savers. However, private wealth effects have an important effect on the *timing* of aggregate consumption. Because savers act according to the permanent income hypothesis, they smooth out the impact of the change in wealth by adjusting consumption in all periods by the same amount. In contrast, borrowers' consumption moves with

[39] The definition of the GE-SE incorporates changes in inflation induced by changes in the borrowers' consumption. At the expense of some more notation, an alternative definition where the GE-SE incorporates changes in inflation induced only by the savers' Hicksian demand generates similar results.

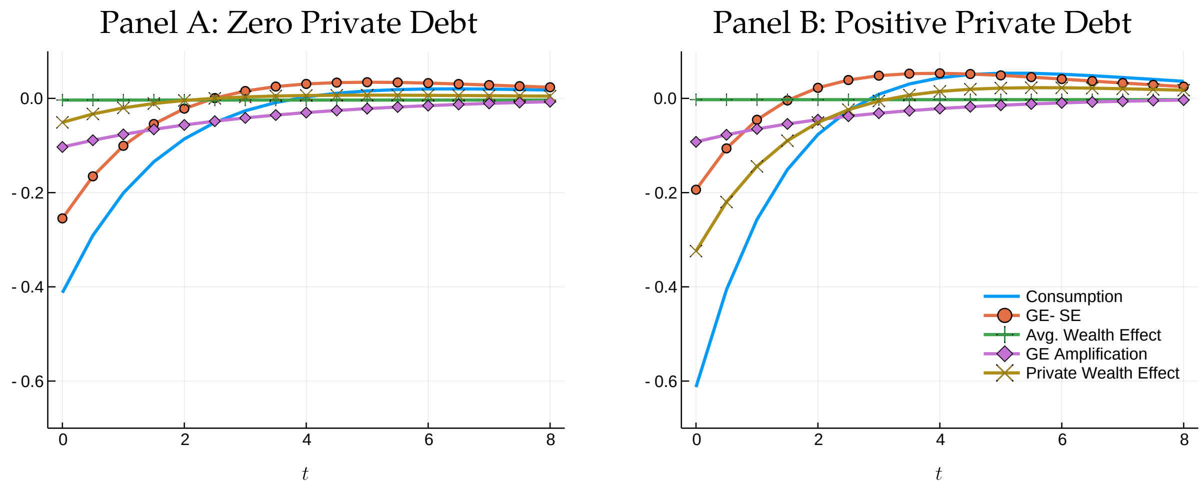

FIGURE 6: Consumption decomposition in a heterogeneous-agent economy

Calibration: $\sigma = 1$, $\kappa = 0.09$, $\rho = 0.01$, $\chi_c = 1$, $\omega = 1/6$, $\alpha = 0.33$, and $\tau = 0.25$. Private debt is set to 40% of GDP and public debt is set to 75% of GDP. Half-life of nominal interest rate is four months. Present value of fiscal transfers is equal to zero.

the cost of servicing the debt. The net effect on consumption depends on how the private wealth effect on borrowers deviates from the wealth effect on savers; that is, it depends on how interest rates deviate from its time-series average. Therefore, the importance of private wealth effects is not determined by the level of interest rates, but instead by the extent to which interest rates are back-loaded or front-loaded.

Private assets also affect the transmission mechanism of monetary policy through its impact on the GE amplification. The presence of compounding in the Euler equation raises the GE multiplier on impact, amplifying the effect of variations in C. Hence, either by creating private wealth effects or by raising the GE amplification, private assets have the potential to raise the impact of nominal interest rate changes on consumption. Figure 6 shows the decomposition (21) for a calibrated example in the cases of zero private debt (Panel A) and positive private debt (Panel B). Consistent with the evidence in Section 3, where a large part of the adjustment came from the revaluation of government bonds, we focus on the case with long-term bonds and zero fiscal transfers. The comparison between the two panels shows that private liquidity substantially amplifies the response to monetary policy shocks. The initial impact on consumption is 50% larger when we introduce private debt. The amplification of the consumption response is explained almost entirely by a larger private wealth effect. Without private debt, roughly 60% of the change in consumption is explained by the GE-SE, and the remaining 40% is explained by a combination of wealth effects and GE amplification. As we introduce private debt, the importance of the GE-SE declines to about 30%, and the private wealth effect explains about 53% of the total consumption response.

An important implication of Proposition 6 is that the private wealth effect is independent of C. From the analysis of Section 2.4, we know that, generically, any value of C must be supported in equilibrium by an appropriate fiscal response or by a revaluation of public debt. We have shown in Section 3 that the quantitative performance of our DSGE model was based on values of the fiscal response much stronger than those observed in the data. In the absence of such movements in fiscal variables, the model was unable to match the dynamics of consumption. The results in Figure 6 indicate that private wealth effects have the potential to create strong responses to monetary policy without relying on large values of C or counterfactual fiscal policy. We next consider the extent to which introducing private wealth effects in our DSGE model can improve its quantitative performance.

4.2 Quantitative importance of private wealth effects

The economy is an extension of the RANK model of Section 2 to incorporate heterogeneous agents. The supply side of the two economies is isomorphic. The difference is on the demand side. We now assume that there are two types of households: a fraction $1-\omega$ of savers (denoted by s) and a fraction ω of borrowers (denoted by b). The two types differ in their discount factor, which satisfies $\beta^b < \beta^s \equiv \beta$. We study the limiting case $\beta^b \to \beta$.

Because borrowers are more impatient than savers, the steady state of the economy features indebted households. We follow Benigno et al. (2019) and assume that the households face an idiosyncratic interest rate, which depends on their debt level and on the aggregate level of indebtedness in the economy. In particular, we assume that while savers' interest rate is $1+i_t^s = 1+i_t$ for all t, the rate faced by borrowers satisfies

$$1+i_t^{b,j} = (1+i_t)\phi\left(\frac{d_t^j}{\bar{d}}, \frac{d_t}{\bar{d}}\right), \tag{22}$$

where $i_t^{b,j}$ is the interest rate faced by borrower j, d_t^j is borrower's j level of debt, $d_t =$

$\frac{\int_0^{1-\chi} d_t^j dj}{1-\chi}$ is aggregate borrowers' debt, $\overline{d}$ is a debt threshold (and the steady-state level of borrowers' debt), and $\phi(\cdot,\cdot)$ is an increasing function with $\phi(x,x) = 1$ for all $x \leq 1$, $\phi_1(1,1) = 0$ and $\phi_2(1,1) > 0$. We assume that the interest rate differential represents a profit taken by unmodeled financial intermediaries.[40] Note that because of their impatience, borrowers never invest in capital, so, in steady state, their capital holdings are zero. Moreover, we assume that all profits of the firms and financial intermediaries accrue to the savers. The model of Section 4.1 is a special case in which $\phi_2(1,1) \to \infty$. The flexibility of equation (22) allows the model to better match the hump-shaped response of consumption to a monetary shock.

We estimate the model using impulse response matching techniques that incorporate fiscal data in the estimation. We drop the Taylor rule and feed the model with the observed path for the nominal interest rate. The TANK model has two important parameters that need to be calibrated: the fraction of borrowers in the economy and the household debt-to-GDP ratio. We set the fraction of borrowers to $1/6$ and the households' debt-to-GDP ratio to 40%. As a reference, Kaplan et al. (2014) find that around one third of U.S. households are hand-to-mouth, while total household debt in the U.S. is close to 80% of GDP. Since our model cannot capture all the nuances of debt and household characteristics (short- versus long-term debt, hand-to-mouth agents with different levels of illiquid assets), we chose a calibration that is conservative in terms of the private wealth effects it generates.[41] The estimated parameter values are reported in Appendix E.

Figure 7 shows the results. The fit of the model improves considerably with respect to the standard RANK. A contractionary monetary shock generates a recession and a drop in consumption and inflation. The change in the interest rate has a strong effect on borrowers' consumption, which cannot be completely smoothed because of the sensitivity of the borrowing cost to the debt level. This effect offsets the absence of wealth effects coming from government transfers, improving the quantitative performance of the model when fiscal variables are set to match the data.[42]

[40] See Benigno et al. (2019) for a detailed derivation.

[41] We performed several robustness checks at different levels of the fraction of borrowers and debt-to-GDP ratio, and the results do not change substantially, either in terms of the estimated parameter values or of the corresponding IRFs to a monetary shock.

[42] Similarly to the IRFs in RANK, the TANK model has difficulty matching the impulse response of con-

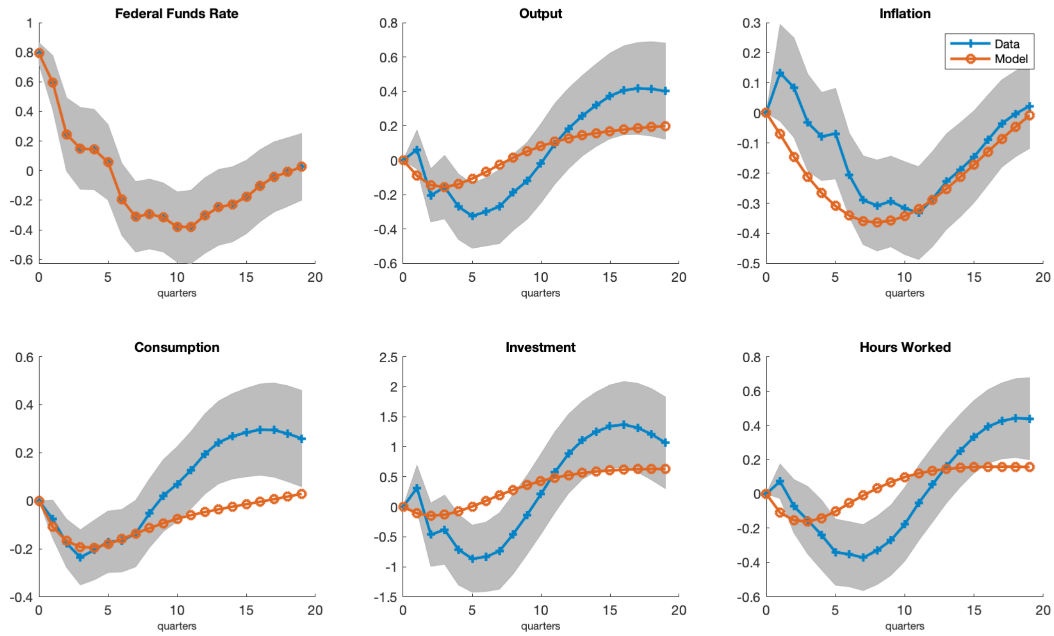

FIGURE 7: Heterogeneous Agent Model impulse response functions to a monetary shock. Gray area represents 95% confidence intervals.

Another important feature of Figure 7 is the failure of the model to generate a substantial drop in investment, which in turn attenuates the overall effect on output. The reason for this result is related to the nature of private wealth effects. An increase in the nominal interest rate generates a redistribution from borrowers to savers. Because borrowers' MPC is larger than savers', overall consumption drops. What do savers do with their increased wealth? Since they do not consume, they invest. Absent strong government-induced wealth effects, the higher interest rate and the induced recession are not enough to generate a drop in investment. Thus, while a redistribution from high to low MPC agents amplifies the effects of a monetary shock on consumption, that same redistribution dampens its effect on investment. This result suggests that a combination of stronger wealth effects and financial frictions that constrain investment might be needed to reconcile the New Keynesian model with the data.

sumption 10 quarters after the monetary shock.

5 Conclusion

In this paper, we provided new analytical tools to understand the role of wealth effects on the transmission mechanism of monetary policy, as well as to quantify their importance in modern RANK and HANK models. We provided a decomposition of the equilibrium consumption response into a general equilibrium substitution effect, a wealth effect, and a GE amplification. Our results showed that the RANK model is unable to generate strong enough wealth effects when it is required to match the observed path of fiscal and monetary variables jointly. Finally, we constructed a heterogeneous agent model that is able to generate the level of wealth effects observed in the data.

Our analysis also uncovered a limitation of HANK models: they may fail in matching the response of investment. Since a contractionary monetary policy shock tends to redistribute towards savers, the response of investment tends to be mitigated related to a RANK model that generates the same level of wealth effect. Future research should focus on understanding how the redistribution channel interacts with investment decisions.

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Appendix: For Online Publication

A Proofs

Proof of Lemma 1.

Integrating the Euler equation between 0 and t, we get

$$c_t = c_0 + \sigma^{-1} \int_0^t (i_s - \pi_s - \rho)\, ds \tag{23}$$

Multiplying both sides by $e^{-\rho t}$ and integrating between 0 and infinity, we get

$$\int_0^\infty e^{-\rho t} c_t dt = \frac{c_0}{\rho} + \frac{\sigma^{-1}}{\rho} \int_0^\infty e^{-\rho t} (i_t - \pi_t - \rho)\, dt$$

Define $C \equiv \rho \int_0^\infty e^{-\rho t} c_t dt$. Then, we can rewrite the expression above as

$$C = c_0 + \sigma^{-1} \int_0^\infty e^{-\rho t} (i_t - \pi_t - \rho)\, dt$$

or

$$c_0 = C - \sigma^{-1} \int_0^\infty e^{-\rho t} (i_t - \pi_t - \rho)\, dt$$

Replacing this expression in (23), we get

$$c_t = \sigma^{-1} \int_0^t (i_s - \pi_s - \rho)\, ds - \sigma^{-1} \int_0^\infty e^{-\rho t} (i_t - \pi_t - \rho)\, dt + C \tag{24}$$

Moreover, integrating the Phillips curve forward, we get (and ruling out explosive paths)

$$\pi_t = \kappa \int_t^\infty e^{-\rho(s-t)} c_s ds \tag{25}$$

Thus, any solution of (1)-(2) is a solution to (23)-(25) together with some value for C. Imposing that $\int_0^\infty e^{-\rho t} c_t dt = 0$ implies that $C = 0$, and (24)-(25) collapses to (10)-(11).

By solving the system (10)-(11) it is straightforward to see that the solution is unique.

■

Proof of Proposition 1.

Let's write consumption as the sum of c_t^S and other terms

$$c_t = \underbrace{c_t^S - \sigma^{-1}\int_0^t \left(\pi_s - \pi_s^S\right) ds + \sigma^{-1}\int_0^\infty e^{-\rho s}\left(\pi_s - \pi_s^S\right) ds}_{=c_t^H} + C$$

Plugging into the Phillips Curve

$$\pi_t = \underbrace{\kappa\int_t^\infty e^{-\rho(s-t)} c_s^S ds}_{=\pi_t^S} - \sigma^{-1}\kappa\int_t^\infty e^{-\rho(s-t)}\int_0^s \left(\pi_z - \pi_z^S\right) dz ds +$$
$$\sigma^{-1}\kappa\int_t^\infty e^{-\rho(s-t)} ds\int_0^\infty e^{-\rho s}\left(\pi_s - \pi_s^S\right) ds + \kappa\int_t^\infty e^{-\rho(s-t)} ds C$$

and after some algebra

$$\pi_t = \pi_t^S - \sigma^{-1}\frac{\kappa}{\rho}\int_0^t \left(\pi_z - \pi_z^S\right) dz - \sigma^{-1}\frac{\kappa}{\rho}e^{\rho t}\int_t^\infty e^{-\rho z}\left(\pi_z - \pi_z^S\right) dz +$$
$$\sigma^{-1}\frac{\kappa}{\rho}\int_0^\infty e^{-\rho s}\left(\pi_s - \pi_s^S\right) ds + \frac{\kappa}{\rho}C$$

Differentiating with respect to time, we get

$$\dot{\pi}_t = \dot{\pi}_t^S - \sigma^{-1}\kappa e^{\rho t}\int_t^\infty e^{-\rho z}\left(\pi_z - \pi_z^S\right) dz$$

Differentiating with respect to time once more, we get

$$\ddot{\pi}_t = \ddot{\pi}_t^S - \sigma^{-1}\kappa\rho e^{\rho t}\int_t^\infty e^{-\rho z}\left(\pi_z - \pi_z^S\right) dz + \sigma^{-1}\kappa\left(\pi_t - \pi_t^S\right)$$

which can be rewritten as

$$\ddot{\pi}_t - \rho\dot{\pi}_t - \sigma^{-1}\kappa\pi_t = \ddot{\pi}_t^S - \rho\dot{\pi}_t^S - \sigma^{-1}\kappa\pi_t^S$$

It is straightforward to see that a solution to the non-homogeneous equation is $\pi_t = \pi_t^S$. For the homogeneous solution, guess that it is given by $\pi_t = e^{\omega t}$, so that $\dot{\pi}_t = \omega e^{\omega t}$ and $\ddot{\pi}_t = \omega^2 e^{\omega t}$. Plugging in

$$\omega^2 - \rho\omega - \sigma^{-1}\kappa = 0$$

Hence

$$\overline{\omega} = \frac{\rho + \sqrt{\rho^2 + 4\sigma^{-1}\kappa}}{2} > 0 \quad \text{and} \quad \underline{\omega} = \frac{\rho - \sqrt{\rho^2 + 4\sigma^{-1}\kappa}}{2} < 0$$

Thus, the general solution is

$$\pi_t = \pi_t^S + c_1 e^{\overline{\omega} t} + c_2 e^{\underline{\omega} t}$$

for some constants c_1 and c_2. Since we are focusing on bounded solutions, $c_1 = 0$. Plugging into the original expression for π_t, we get

$$\pi_t = \pi_t^S - \sigma^{-1}\frac{\kappa}{\rho}\int_0^t \left(\pi_z^S + c_2 e^{\underline{\omega} z} - \pi_z^S\right) dz - \sigma^{-1}\frac{\kappa}{\rho}\int_t^\infty e^{-\rho(z-t)}\left(\pi_z^S + c_2 e^{\underline{\omega} z} - \pi_z^S\right) dz +$$
$$\sigma^{-1}\frac{\kappa}{\rho}\int_0^\infty e^{-\rho s}\left(\pi_s^S + c_2 e^{\underline{\omega} s} - \pi_s^S\right) ds + \frac{\kappa}{\rho}C$$

After some algebra, we get

$$\pi_t = \pi_t^S - c_2 + c_2 e^{\underline{\omega} t} + \frac{\kappa}{\rho}C$$

Using that $\pi_t = \pi_t^S + c_2 e^{\underline{\omega} t}$ and matching coefficients, we get

$$c_2 e^{\underline{\omega} t} = -c_2 + c_2 e^{\underline{\omega} t} + \frac{\kappa}{\rho}C$$
$$c_2 = \frac{\kappa}{\rho}C$$

Plugging in $\pi_t = \pi_t^S + \frac{\kappa}{\rho} e^{\underline{\omega} t} C$ into c_t, we get

$$c_t = c_t^S - \sigma^{-1}\frac{\kappa}{\rho}\int_0^t e^{\underline{\omega} s} ds C + \sigma^{-1}\frac{\kappa}{\rho}\int_0^\infty e^{-\rho s} e^{\underline{\omega} s} ds C + C$$

or

$$c_t = c_t^S + \frac{\overline{\omega}}{\rho} e^{\underline{\omega} t} C$$

■

Proof of Corollary 1.1.

Immediate from the fact that, given $\{i_t\}_{t=0}^\infty$, $\{c_t^S, \pi_t^S\}_{t=0}^\infty$ is unique and $\frac{\overline{\omega}}{\rho} e^{\underline{\omega} t}$ depends only on the parameters of the model.

■

Proof of Lemma 2.

Consider an economy characterized by the following system of equations:

$$\dot{c}_t = \sigma^{-1}(i_t - \pi_t - \rho)$$
$$\dot{\pi}_t = \rho \pi_t + \kappa c_t$$
$$i_t = \rho + \phi \pi_t + \varepsilon_t$$

where $\phi \in \left(1, 1 + \frac{\rho^2}{4\kappa\sigma^{-1}}\right]$.The solution to this system is given by

$$\begin{aligned}
c_t^* &= -\frac{\sigma^{-1}}{\overline{\delta} - \underline{\delta}} \int_t^\infty \left(\overline{\delta} e^{-\underline{\delta}(s-t)} - \underline{\delta} e^{-\overline{\delta}(s-t)}\right) \varepsilon_s ds \\
\pi_t^* &= -\frac{\sigma^{-1}\kappa}{\overline{\delta} - \underline{\delta}} \int_t^\infty \left(e^{-\underline{\delta}(s-t)} - e^{-\overline{\delta}(s-t)}\right) \varepsilon_s ds \\
i_t^* &= \rho - \phi \frac{\sigma^{-1}\kappa}{\overline{\delta} - \underline{\delta}} \int_t^\infty \left(e^{-\underline{\delta}(s-t)} - e^{-\overline{\delta}(s-t)}\right) \varepsilon_s ds + \varepsilon_t.
\end{aligned}$$

where

$$\overline{\delta} = \frac{\rho + \sqrt{\rho^2 + 4(1-\phi)\kappa\sigma{-1}}}{2}, \quad \underline{\delta} = \frac{\rho - \sqrt{\rho^2 + 4(1-\phi)\kappa\sigma{-1}}}{2}.$$

We want to show that

$$-\frac{\sigma^{-1}}{\overline{\omega} - \underline{\omega}} \int_t^\infty \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right) (i_s^* - \rho)\, ds = -\frac{\sigma^{-1}}{\overline{\delta} - \underline{\delta}} \int_t^\infty \left(\overline{\delta} e^{-\underline{\delta}(s-t)} - \underline{\delta} e^{-\overline{\delta}(s-t)}\right) \varepsilon_s ds.$$

We will work with the LHS of the previous equality. Plugging in the solution for $\{i_t^*\}_{t=0}^\infty$, we get

$$\begin{aligned}
-\frac{\sigma^{-1}}{\overline{\omega} - \underline{\omega}} \int_t^\infty \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right) (i_s^* - \rho)\, ds = -\frac{\sigma^{-1}}{\overline{\omega} - \underline{\omega}} \int_t^\infty \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right) \varepsilon_s ds + \\
\frac{\sigma^{-1}}{\overline{\omega} - \underline{\omega}} \phi \frac{\sigma^{-1}\kappa}{\overline{\delta} - \underline{\delta}} \int_t^\infty \int_s^\infty \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right) \left(e^{-\underline{\delta}(z-s)} - e^{-\overline{\delta}(z-s)}\right) \varepsilon_z dz ds \quad (26)
\end{aligned}$$

Consider the last integral

$$\int_t^\infty \int_s^\infty \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right) \left(e^{-\underline{\delta}(z-s)} - e^{-\overline{\delta}(z-s)}\right) \varepsilon_z dz ds$$

Let's study the different terms separately. Consider first

$$\int_t^\infty \int_s^\infty \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right) \left(e^{-\underline{\delta}(z-s)} - e^{-\overline{\delta}(z-s)}\right) \varepsilon_z dz ds$$
$$= \int_t^\infty \int_t^z \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right) \left(e^{-\underline{\delta}(z-s)} - e^{-\overline{\delta}(z-s)}\right) \varepsilon_z ds dz$$
$$= \int_t^\infty \int_t^z \left(\overline{\omega} e^{\underline{\omega} t} e^{-\underline{\delta} z} e^{(\underline{\delta}-\underline{\omega})s} - \overline{\omega} e^{\underline{\omega} t} e^{-\overline{\delta} z} e^{\left(\overline{\delta}-\underline{\omega}\right)s} - \underline{\omega} e^{\overline{\omega} t} e^{-\underline{\delta} z} e^{(\underline{\delta}-\overline{\omega})s} + \underline{\omega} e^{\overline{\omega} t} e^{-\overline{\delta} z} e^{\left(\overline{\delta}-\overline{\omega}\right)s}\right) \varepsilon_z ds dz$$
$$= \int_t^\infty \left(\frac{\overline{\delta}-\underline{\delta}}{(\underline{\delta}-\underline{\omega})\left(\overline{\delta}-\underline{\omega}\right)} \left(\overline{\omega} e^{-\underline{\omega}(z-t)} - \underline{\omega} e^{-\overline{\omega}(z-t)}\right) + \frac{\underline{\omega}\left(\underline{\delta}-\underline{\omega}\right) - \overline{\omega}\left(\underline{\delta}-\overline{\omega}\right)}{\left(\underline{\delta}-\overline{\omega}\right)\left(\underline{\delta}-\underline{\omega}\right)} e^{-\underline{\delta}(z-t)} - \right.$$
$$\left. \frac{\underline{\omega}\left(\overline{\delta}-\underline{\omega}\right) - \overline{\omega}\left(\overline{\delta}-\overline{\omega}\right)}{\left(\overline{\delta}-\overline{\omega}\right)\left(\overline{\delta}-\underline{\omega}\right)} e^{-\overline{\delta}(z-t)}\right) \varepsilon_z dz$$

Note that $\left(\underline{\delta}-\underline{\omega}\right)\left(\overline{\delta}-\underline{\omega}\right) = \left(\underline{\delta}-\overline{\omega}\right)\left(\overline{\delta}-\overline{\omega}\right) = \phi\kappa\sigma^{-1}$ and $\left(\underline{\delta}-\overline{\omega}\right)\left(\underline{\delta}-\underline{\omega}\right) = \left(\overline{\delta}-\overline{\omega}\right)\left(\overline{\delta}-\underline{\omega}\right) = -\phi\kappa\sigma^{-1}$. Hence

$$\int_t^\infty \int_s^\infty \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right) \left(e^{-\underline{\delta}(z-s)} - e^{-\overline{\delta}(z-s)}\right) \varepsilon_z dz ds =$$
$$\frac{\overline{\delta}-\underline{\delta}}{\phi\kappa\sigma^{-1}} \int_t^\infty \left(\overline{\omega} e^{-\underline{\omega}(z-t)} - \underline{\omega} e^{-\overline{\omega}(z-t)}\right) \varepsilon_z dz -$$
$$\int_t^\infty \left(\frac{\underline{\omega}\left(\underline{\delta}-\underline{\omega}\right) - \overline{\omega}\left(\underline{\delta}-\overline{\omega}\right)}{\phi\kappa\sigma^{-1}} e^{-\underline{\delta}(z-t)} - \frac{\underline{\omega}\left(\overline{\delta}-\underline{\omega}\right) - \overline{\omega}\left(\overline{\delta}-\overline{\omega}\right)}{\phi\kappa\sigma^{-1}} e^{-\overline{\delta}(z-t)}\right) \varepsilon_z dz$$

Moreover, note that $\underline{\omega}\left(\underline{\delta}-\underline{\omega}\right) - \overline{\omega}\left(\underline{\delta}-\overline{\omega}\right) = \left(\overline{\omega}-\underline{\omega}\right)\overline{\delta}$ and $\underline{\omega}\left(\overline{\delta}-\underline{\omega}\right) - \overline{\omega}\left(\overline{\delta}-\overline{\omega}\right) = \left(\overline{\omega}-\underline{\omega}\right)\underline{\delta}$, hence

$$\int_t^\infty \int_s^\infty \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right) \left(e^{-\underline{\delta}(z-s)} - e^{-\overline{\delta}(z-s)}\right) \varepsilon_z dz ds =$$
$$\frac{\overline{\delta}-\underline{\delta}}{\phi\kappa\sigma^{-1}} \int_t^\infty \left(\overline{\omega} e^{-\underline{\omega}(z-t)} - \underline{\omega} e^{-\overline{\omega}(z-t)}\right) \varepsilon_z dz - \frac{\overline{\omega}-\underline{\omega}}{\phi\kappa\sigma^{-1}} \int_t^\infty \left(\overline{\delta} e^{-\underline{\delta}(z-t)} - \underline{\delta} e^{-\overline{\delta}(z-t)}\right) \varepsilon_z dz$$

Plugging back into (26), we get

$$c_t = \frac{\sigma^{-1}}{\overline{\omega}-\underline{\omega}} \int_t^\infty \left(\overline{\omega} e^{-\underline{\omega}(z-t)} - \underline{\omega} e^{-\overline{\omega}(z-t)}\right) \varepsilon_z dz - \frac{\sigma^{-1}}{\overline{\delta}-\underline{\delta}} \int_t^\infty \left(\overline{\delta} e^{-\underline{\delta}(z-t)} - \underline{\delta} e^{-\overline{\delta}(z-t)}\right) \varepsilon_z dz -$$
$$\frac{\sigma^{-1}}{\overline{\omega}-\underline{\omega}} \int_t^\infty \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right) \varepsilon_s ds$$

and hence

$$c_t = -\frac{\sigma^{-1}}{\overline{\delta}-\underline{\delta}} \int_t^\infty \left(\overline{\delta} e^{-\underline{\delta}(s-t)} - \underline{\delta} e^{-\overline{\delta}(s-t)}\right) \varepsilon_s ds.$$

Next, we show that $c_t = -\frac{\sigma^{-1}}{\overline{\omega}-\underline{\omega}} \int_t^\infty \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right)(i_s^* - \rho)\, ds$ is the unique purely forward-looking solution of the system (1)-(2). Recall that the solution to the system can be written as

$$c_t = c_t^S + \frac{\overline{\omega}}{\rho} e^{\underline{\omega}t} C \tag{27}$$

with

$$c_t^S = \frac{\sigma^{-1}\overline{\omega}}{\overline{\omega}-\underline{\omega}} e^{\underline{\omega}t} \int_0^t \left(e^{-\underline{\omega}s} - e^{-\overline{\omega}s}\right)(i_s - \rho)\, ds + \frac{\sigma^{-1}}{\overline{\omega}-\underline{\omega}} \left(\underline{\omega} e^{\overline{\omega}t} - \overline{\omega} e^{\underline{\omega}t}\right) \int_t^\infty e^{-\overline{\omega}s}(i_s - \rho)\, ds$$

Since C is independent of t and equal to zero when $i_t = \rho$ for all t, it has to take the following form:

$$C = \int_0^\infty \Omega_s (i_s - \rho)\, ds$$

Plugging c_t^H and C into (27), we get

$$\begin{aligned} c_t = \frac{\sigma^{-1}\overline{\omega}}{\overline{\omega}-\underline{\omega}} e^{\underline{\omega}t} \int_0^t \left(e^{-\underline{\omega}s} - e^{-\overline{\omega}s}\right)(i_s - \rho)\, ds + \frac{\sigma^{-1}}{\overline{\omega}-\underline{\omega}} \left(\underline{\omega} e^{\overline{\omega}t} - \overline{\omega} e^{\underline{\omega}t}\right) \int_t^\infty e^{-\overline{\omega}s}(i_s - \rho)\, ds + \\ \frac{\overline{\omega}}{\rho} e^{\underline{\omega}t} \int_0^t \Omega_s (i_s - \rho)\, ds + \frac{\overline{\omega}}{\rho} e^{\underline{\omega}t} \int_t^\infty \Omega_s (i_s - \rho)\, ds \end{aligned}$$

where we divided the integral in C into a backward-looking and a forward-looking term. Combining terms, we get

$$\begin{aligned} c_t = \overline{\omega} e^{\underline{\omega}t} \int_0^t \left(\frac{\sigma^{-1}}{\overline{\omega}-\underline{\omega}} e^{-\underline{\omega}s} - \frac{\sigma^{-1}}{\overline{\omega}-\underline{\omega}} e^{-\overline{\omega}s} + \frac{1}{\rho}\Omega_s\right)(i_s - \rho)\, ds + \\ \frac{\sigma^{-1}}{\overline{\omega}-\underline{\omega}} \left(\underline{\omega} e^{\overline{\omega}t} - \overline{\omega} e^{\underline{\omega}t}\right) \int_t^\infty e^{-\overline{\omega}s}(i_s - \rho)\, ds + \frac{\overline{\omega}}{\rho} e^{\underline{\omega}t} \int_t^\infty \Omega_s (i_s - \rho)\, ds \end{aligned} \tag{28}$$

This expression is purely forward-looking if and only if $\frac{\sigma^{-1}}{\overline{\omega}-\underline{\omega}} e^{-\underline{\omega}s} - \frac{\sigma^{-1}}{\overline{\omega}-\underline{\omega}} e^{-\overline{\omega}s} + \frac{1}{\rho}\Omega_s = 0$ almost surely, or

$$\Omega_s = -\frac{\sigma^{-1}\rho}{\overline{\omega}-\underline{\omega}} \left(e^{-\underline{\omega}s} - e^{-\overline{\omega}s}\right)$$

Plugging this expression into (28), we get

$$c_t = -\frac{\sigma^{-1}}{\overline{\omega}-\underline{\omega}} \int_t^\infty \left(\overline{\omega} e^{-\underline{\omega}(s-t)} - \underline{\omega} e^{-\overline{\omega}(s-t)}\right)(i_s - \rho)\, ds$$

and

$$C = -\frac{\sigma^{-1}\rho}{\overline{\omega}-\underline{\omega}} \int_0^\infty \left(e^{-\underline{\omega}s} - e^{-\overline{\omega}s}\right)(i_s - \rho)\, ds$$

Note that C is finite if and only if $i_t = \mathcal{O}\left(e^{-\theta t}\right)$ for some $\theta > |\underline{\omega}|$. This condition holds since the sequence of shocks decays sufficiently fast.

■

Proof of Proposition 2.

Immediate from the proof of Proposition 1.

■

Proof of Lemma 3.

We have

$$\lim_{t\to\infty} c_t = \lim_{t\to\infty} c_t^S + \lim_{t\to\infty} \frac{\overline{\omega}}{\rho} e^{\underline{\omega} t} C$$

and

$$\lim_{t\to\infty} \pi_t = \lim_{t\to\infty} \pi_t^S + \lim_{t\to\infty} \frac{\overline{\kappa}}{\rho} e^{\underline{\omega} t} C$$

Since $\underline{\omega} < 0$, the second term of both equations is equal to zero. Thus, we only need to determine $\lim_{t\to\infty} c_t^S$ and $\lim_{t\to\infty} \pi_t^S$.

We have

$$c_t^S = \sigma^{-1} \int_0^t (i_s - \rho)\, ds - \sigma^{-1} \int_0^\infty e^{-\rho s} (i_s - \rho)\, ds - \sigma^{-1} \int_0^t \pi_s^S ds + \sigma^{-1} \int_0^\infty e^{-\rho s} \pi_s^S ds$$

$$\pi_t^S = \kappa \int_t^\infty e^{-\rho(s-t)} c_s^S ds$$

Plugging c_t^H into π_t^H, we get

$$\pi_t^S = \kappa \int_t^\infty e^{-\rho(s-t)} \left(\sigma^{-1} \int_0^s (i_z - \rho)\, dz - \sigma^{-1} \int_0^\infty e^{-\rho z} (i_z - \rho)\, dz - \sigma^{-1} \int_0^s \pi_z^S dz + \right.$$
$$\left. \sigma^{-1} \int_0^\infty e^{-\rho z} \pi_z^S dz \right) ds$$

After some algebra

$$\pi_t^S = \sigma^{-1} \frac{\kappa}{\rho} \int_0^t (i_z - \rho)\, dz + \sigma^{-1} \frac{\kappa}{\rho} e^{\rho t} \int_t^\infty e^{-\rho z} (i_z - \rho)\, dz - \sigma^{-1} \frac{\kappa}{\rho} \int_0^\infty e^{-\rho z} (i_z - \rho)\, dz -$$
$$\sigma^{-1} \frac{\kappa}{\rho} \int_0^t \pi_z^S dz - \sigma^{-1} \frac{\kappa}{\rho} e^{\rho t} \int_t^\infty e^{-\rho z} \pi_z^S dz + \sigma^{-1} \frac{\kappa}{\rho} \int_0^\infty e^{-\rho z} \pi_z^S dz$$

Differentiating with respect to time, we get

$$\dot{\pi}_t^S = \sigma^{-1} \kappa e^{\rho t} \int_t^\infty e^{-\rho z} (i_z - \rho)\, dz - \sigma^{-1} \kappa e^{\rho t} \int_t^\infty e^{-\rho z} \pi_z^S dz$$

Differenting with respect to time once more, we get

$$\ddot{\pi}_t^S - \rho\dot{\pi}_t^S - \sigma^{-1}\kappa\pi_t^S = -\sigma^{-1}\kappa\left(i_t - \rho\right)$$

The solution to this second-order differential equation is

$$\pi_t^S = \frac{\sigma^{-1}\kappa}{\overline{\omega} - \underline{\omega}} e^{\underline{\omega}t} \int_0^t (e^{-\underline{\omega}s} - e^{-\overline{\omega}s})\left(i_s - \rho\right) ds + \frac{\sigma^{-1}\kappa}{\overline{\omega} - \underline{\omega}} (e^{\overline{\omega}t} - e^{\underline{\omega}t}) \int_t^\infty e^{-\overline{\omega}s}\left(i_s - \rho\right) ds$$

Thus

$$\begin{aligned}
\lim_{t\to\infty} \pi_t^S &= \frac{\sigma^{-1}\kappa}{\overline{\omega} - \underline{\omega}} \lim_{t\to\infty} \left[\frac{\int_0^t e^{-\underline{\omega}s}(i_s - \rho)ds}{e^{-\underline{\omega}t}} + \frac{\int_t^\infty e^{-\overline{\omega}s}(i_s - \rho)ds}{e^{-\overline{\omega}t}} \right] \\
&= \frac{\sigma^{-1}\kappa}{\overline{\omega} - \underline{\omega}} \lim_{t\to\infty} \left[-\frac{e^{-\underline{\omega}t}(i_t - \rho)}{\underline{\omega}e^{-\underline{\omega}t}} + \frac{e^{-\overline{\omega}t}(i_t - \rho)}{\overline{\omega}e^{-\overline{\omega}t}} \right] \\
&= 0
\end{aligned}$$

Plugging into c_t^S, we get

$$c_t^S = \sigma^{-1} \frac{\overline{\omega}}{\overline{\omega} - \underline{\omega}} e^{\underline{\omega}t} \int_0^t \left(e^{-\underline{\omega}z} - e^{-\overline{\omega}z} \right) \left(i_z - \rho\right) dz - \sigma^{-1} \frac{\overline{\omega}e^{\underline{\omega}t} - \underline{\omega}e^{\overline{\omega}t}}{\overline{\omega} - \underline{\omega}} \int_t^\infty e^{-\overline{\omega}z}\left(i_z - \rho\right) dz$$

Thus,

$$\begin{aligned}
\lim_{t\to\infty} c_t^S &= \frac{\sigma^{-1}}{\overline{\omega} - \underline{\omega}} \lim_{t\to\infty} \left[\overline{\omega} \frac{\int_0^t e^{-\underline{\omega}z}\left(i_z - \rho\right) dz}{e^{-\underline{\omega}t}} + \underline{\omega} \frac{\int_t^\infty e^{-\overline{\omega}z}\left(i_z - \rho\right) dz}{e^{-\overline{\omega}t}} \right] \\
&= \frac{\sigma^{-1}}{\overline{\omega} - \underline{\omega}} \lim_{t\to\infty} \left[-\frac{\overline{\omega}}{\underline{\omega}} \frac{e^{-\underline{\omega}t}\left(i_t - \rho\right)}{e^{-\underline{\omega}t}} + \frac{\underline{\omega}}{\overline{\omega}} \frac{e^{-\overline{\omega}t}\left(i_t - \rho\right)}{e^{-\overline{\omega}t}} \right] \\
&= 0
\end{aligned}$$

■

Proof of Proposition 3.

Immediate from equation (13).

■

Proof of Lemma 4. Average consumption in the presence of long-term bonds and zero transfers can be written as

$$C = \rho \int_0^\infty e^{-\rho t} \left[(1-\tau)y_t + b(i_t - \pi_t - \rho)\right] dt - \rho b \int_0^\infty e^{-(\rho+m)t}(i_t - \rho)dt.$$

Using the spending-income spiral and the spending-inflation spiral, we can rewrite this as

$$C = \frac{\rho}{\tau - \sigma\underline{\omega}b}\left(\int_0^\infty e^{-\rho t} b(i_t - \pi_t^S - \rho)dt - b\int_0^\infty e^{-(\rho+m)t}(i_t - \rho)dt\right)$$

Replacing with the expression for π_t^S, and after some algebra,

$$C = \frac{\rho}{\tau - \sigma\underline{\omega}b}\left(\int_0^\infty e^{-\overline{\omega}t} b(i_t - \rho)dt - b\int_0^\infty e^{-(\rho+m)t}(i_t - \rho)dt\right)$$

It is immediate to see that $\frac{\partial C}{\partial i_t} \geq 0$ if and only if $m \geq -\underline{\omega}$ and $\frac{\partial C}{\partial i_t} < 0$ otherwise. ■

Proof of Proposition 4.

Let $Y_t = [Y'_{J,t} Y_{P,t'}]'$ denote the N-dimensional of the (log-linearized) endogenous variables, excluding the nominal interest rates. We will treat consumption effectively as a predetermined variable, where the initial condition for consumption is pinned down by C. Then, $Y_{J,t}$ denotes the $(j-1)$-dimensional vector of jump variables except consumption, and $Y_{P,t}$ denotes the $(p+1)$-dimensional vector of predetermined variables in addition to consumption, where $N = j + p$. The dynamics Y_t can be written as

$$Y_{t+1} = AY_t + b(i_t - \rho) \tag{29}$$

where A is a $N \times N$ matrix of coefficients and b is a $N \times 1$ vector.

Given the certainty-equivalence property of linearized models, without loss of generality we consider a perfect foresight dynamic system. Consider next the eigendecomposition of the matrix A

$$A = \underbrace{\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \Lambda_{11} & \mathbf{0} \\ \mathbf{0} & \Lambda_{22} \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{bmatrix}}_{V^{-1}} \tag{30}$$

where V is the matrix of eigenvectors of A and Λ is a diagonal matrix of eigenvalues. For this proof, we need to assume that a version of the Blanchard-Kahn condition is satisfied. We define the *generalized Blanchard-Kahn* condition as follows.

Definition 2. *Let A denote the matrix of coefficients defined in* (29)*. A generalized Blanchard-Kahn condition is satisfied if*

1. *The number of eigenvalues of A with absolute value greater than one equals $p + 1$, where p is the number of predetermined variables in the system.*

2. *The matrix V^{11} defined in equation* (30) *is invertible.*

The condition above coincides with the standard Blanchard-Kahn conditions if we treat consumption as predetermined and the nominal interest rate as exogenous. In what follows, we assume that this condition is satisified.

By adopting the change of coordinates, $\tilde{Y}_t = V^{-1}Y_t$ and $\tilde{b} = V^{-1}b$, we can write the system in decoupled form

$$\tilde{Y}_{t+1} = \Lambda\tilde{Y}_t + \tilde{b}(i_t - \rho)$$

Suppose that all elements in the diagonal of Λ_{11} are larger than one in absolute value. Then, we can write

$$\tilde{Y}_{J,t} = -\Lambda_{11}^{-1}\tilde{b}_J(i_t - \rho) + \Lambda_{11}^{-1}\tilde{Y}_{J,t+1} \Rightarrow \tilde{Y}_{J,t} = -\sum_{k=1}^{\infty}\Lambda_{11}^{-k}\tilde{b}_J(i_{t+k-1} - \rho)$$

Note that we can write $\tilde{Y}_{J,t} = V^{11}Y_{J,t} + V^{12}Y_{P,t}$. Then,

$$Y_{J,t} = -(V^{11})^{-1}V^{12}Y_{P,t} - (V^{11})^{-1}\sum_{k=1}^{\infty}\Lambda_{11}^{-k}\tilde{b}_J(i_{t+k-1} - \rho)$$

assuming that V^{11} is invertible.

Suppose all elements in the diagonal of Λ_{22} are smaller than one in absolute value. Then, we can write

$$\tilde{Y}_{P,t+1} = \Lambda_{22}\tilde{Y}_{P,t} + \tilde{b}_J(i_t - \rho) \Rightarrow \tilde{Y}_{P,t} = \Lambda_{22}^{t}\tilde{Y}_{p,0} - \sum_{k=0}^{t}\Lambda_{22}^{k}\tilde{b}_P(i_{t-k-1} - \rho)$$

Note that we can write $\tilde{Y}_{P,t} = V^{21}Y_{J,t} + V^{22}Y_{P,t}$. Then,

$$\begin{aligned}\left[V^{22} - V^{21}(V^{11})^{-1}V^{12}\right]Y_{p,t} = \Lambda_{22}^{t}\left[V^{22} - V^{21}(V^{11})^{-1}V^{12}\right]Y_{p,0} + V^{21}(V^{11})^{-1}\sum_{k=1}^{\infty}\Lambda_{11}^{-k}\tilde{b}_J(i_{t+k-1} - \rho)\\ - \sum_{k=0}^{t}\Lambda_{22}^{k}\tilde{b}_P(i_{t-k-1} - \rho) - \Lambda_{22}^{t}V^{21}(V^{11})^{-1}\sum_{k=1}^{\infty}\Lambda_{11}^{-k}\tilde{b}_J(i_{k-1} - \rho)\end{aligned}$$

Using $\left[V^{22} - V^{21}(V^{11})^{-1}V^{12}\right] = V_{22}$ by the inverse of a partitioned matrix, then

$$Y_{p,t} = V_{11}\Lambda_{22}^{t}V_{11}^{-1}Y_{p,0} + Z_{p,t}$$

where $Z_{p,t}$ is a function of the sequence of nominal interest rates.

$$Z_{p,t} \equiv V_{11}V^{21}(V^{11})^{-1}\sum_{k=1}^{\infty}\Lambda_{11}^{-k}\tilde{b}_J(i_{t+k-1}-\rho) - V_{11}\sum_{k=0}^{t}\Lambda_{22}^{k}\tilde{b}_P(i_{t-k-1}-\rho) - \\ V_{11}\Lambda_{22}^{t}V^{21}(V^{11})^{-1}\sum_{k=1}^{\infty}\Lambda_{11}^{-k}\tilde{b}_J(i_{k-1}-\rho)$$

Using the fact that the initial condition for all predetermined variables is equal to zero, except for consumption, we can write

$$c_t = \sum_{k=0}^{p+1}\tilde{\nu}_k\lambda_k^t c_0 + Z_{c,t}$$

where λ_k is the k-th element in the diagonal of λ_{22}.

Averaging the equation above over all t and writing c_0 in terms of C, we obtain

$$c_t = \sum_{k=0}^{p+1}\nu_k\lambda_k^t C + \tilde{Z}_{c,t}$$

where $\tilde{Z}_{c,t}$ is a function of the sequence of nominal interest rates, for a given set of coefficients ν_k. Recall that by the properties of the Marshallian demand, $c_t = c_t^H + C$. Moreover, if $C = 0$, then $c_t = c_t^S$ (i.e., the Hicksian demand evaluated at the inflation rate consistent with the Hicksian demand). Thus, by setting $\tilde{Z}_{c,t} = c_t^S$, we obtain the result in the proposition.

Importantly, the properties of the Hicksian demand are not affected by the presence of habit formation. The utility of the household is given by

$$\overline{U} = \sum_{t=0}^{\infty}\beta^t\log\left(C_t - bC_{t-1}\right)$$

Log-linearizing the expression above around the steady state, and assuming $c_{-1} = 0$, we obtain

$$0 = \sum_{t=0}^{\infty}\beta^t\frac{c_t - bc_{t-1}}{1-b} \Rightarrow \sum_{t=0}^{\infty}\beta^t c_t = b\beta\sum_{t=1}^{\infty}\beta^{t-1}c_{t-1}$$

rearranging the expression above

$$\sum_{t=0}^{\infty}\beta^t c_t = 0$$

Hence, average consumption is equal to zero for the Hicksian demand even in the case of preferences with habit.

■

Proof of Proposition 5. We log-linearize the expressions around a steady state in which borrowers are constrained and savers are unconstrained. For simplicity, we focus on the steady state $C_b = C_s = C$, which implies $N_b = N_s = N$. The Euler equation for savers can be written as

$$\dot{c}_{s,t} = \frac{i_t - \pi_t - \rho}{\sigma}$$

The labor supply condition can be written as

$$w_t - p_t = \phi n_{j,t} + \sigma c_{j,t}$$

Log-linearizing the market clearing conditions for consumption and labor, we obtain

$$\chi c_b + (1 - \chi_b) c_s = c_t$$
$$\chi n_b + (1 - \chi_b) n_s = n_t$$

The budget constraint for savers can be written as

$$\overline{b}_s \dot{b}_{s,t} = (i_t - \pi_t - \rho)\overline{b}_s + \rho \overline{b}_s b_{s,t} + (1 - \alpha)(w_t - p_t + n_{s,t}) + \frac{(1 - \tau)c_t - (1 - \alpha)(w_t - p_t + c_t)}{1 - \omega} + T_{s,t} - c_{s,t}$$

The budget constraint for borrowers can be written as

$$c_{b,t} = (1 - \alpha)(w_t - p_t + n_{b,t}) + T_{b,t} - (i_t - \pi_t - \rho)\overline{d}_P$$

where $1 - \alpha \equiv \frac{W}{PY}$ is the labor share, $T_{b,t} \equiv \frac{\tilde{T}_{b,t} - \tilde{T}_b}{Y}$, and $\overline{d}_p \equiv \frac{\overline{D}_p}{Y}$.

Aggregating the labor supply condition, we obtain

$$w_t - p_t = (\phi + \sigma) c_t$$

Suppose the government transfers to borrowers are a function of aggregate consumption

$$T_{b,t} = \psi_b c_t$$

Combining the two previous expressions, and using the labor supply condition to eliminate $n_{b,t}$, we obtain

$$c_{b,t} = \chi_c c_t - \chi_r (i_t - \pi_t - \rho)$$

where

$$\chi_c \equiv \frac{(1-\alpha)(1+\phi^{-1})(\phi+\sigma) - \psi_b}{1+(1-\alpha)\phi^{-1}\sigma}$$
$$\chi_r \equiv \frac{\bar{d}_p}{1+(1-\alpha)\phi^{-1}\sigma}$$

From the market clearing condition for goods, we obtain the consumption of savers

$$c_s = \frac{1-\omega\chi_c}{1-\omega} c_t + \frac{\omega\chi_r}{1-\omega}(i_t - \pi_t - \rho)$$

Combining the expression above with the Euler equation, we obtain

$$\dot{c}_t = \frac{1-\omega}{1-\omega\chi_c}\sigma^{-1}\left(i_t - \pi_t - \rho\right) - \frac{\omega\chi_r}{1-\omega\chi_c}(\dot{i}_t - \dot{\pi}_t)$$

The optimality condition for firms can be written as

$$p_t^*(j) = (\rho + \rho_\delta)\int_t^\infty e^{-(\rho+\rho_\delta)(z-t)} w_z dz$$

Log-linearizing the expression for the price index, we obtain

$$p_t = \rho_\delta \int_{-\infty}^t e^{-\rho_\delta(t-z)} p_z^* dz$$

Differentiating the expression above, we obtain

$$\pi_t = \rho_\delta(p_t^* - p_t)$$

Differentiating the expression again, we obtain

$$\begin{aligned}\dot{\pi}_t &= \rho_\delta(\rho+\rho_\delta)p_t^* - (\rho+\rho_\delta)\rho_\delta w_t - \rho_\delta\pi_t \\ &= \rho\pi_t - (\rho+\rho_\delta)\rho_\delta(w_t - p_t)\end{aligned}$$

Using the expression for the real wage, we obtain the New Keynesian Phillips curve

$$\dot{\pi}_t = \rho\pi_t - \kappa c_t \tag{31}$$

where $\kappa \equiv \rho_\delta(\rho + \rho_\delta)(\phi + \sigma)$.

Combining the expression above with the evolution of consumption, we obtain

$$\begin{aligned}\dot{c}_t &= \frac{1-\omega}{1-\omega\chi_c}\sigma^{-1}\left(i_t - \pi_t - \rho\right) - \frac{\omega\chi_r}{1-\omega\chi_c}(\dot{i}_t - \rho(i_t - \rho) + \rho(i_t - \pi_t - \rho) + \kappa c_t) \\ &= \frac{(1-\omega)\sigma^{-1} - \rho\omega\chi_r}{1-\omega\chi_c}\left(i_t - \pi_t - \rho\right) - \frac{\omega\chi_r}{1-\omega\chi_c}(\dot{i}_t - \rho(i_t - \rho)) - \frac{\omega\chi_r}{1-\omega\chi_c}\kappa c_t\end{aligned}$$

The aggregate Euler equation can then be written as

$$\dot{c}_t = \tilde{\sigma}^{-1}(i_t - \pi_t - \rho) - \delta c_t + v_t$$

where

$$\begin{aligned}\tilde{\sigma}^{-1} &= \frac{(1-\omega)\sigma^{-1} - \rho\omega\chi_r}{1-\omega\chi_c} \\ \delta &= \frac{\omega\chi_r}{1-\omega\chi_c}\kappa \\ v_t &= -\frac{\omega\chi_r}{1-\omega\chi_c}(\dot{i}_t - \rho(i_t - \rho))\end{aligned}$$

Combining the budget constraints for borrowers and savers, we obtain the intertemporal budget constraint

$$\int_0^\infty e^{-\rho t} c_t dt = \int_0^\infty e^{-\rho t}\left[(1-\tau)y_t + \bar{b}(i_t - \pi_t - \rho) + T_t\right] dt$$

where $T_t \equiv \omega T_{b,t} + (1-\omega)T_{s,t}$.

■

Proof of Proposition 6.

This proof consists of two steps. First, we provide a solution to the dynamic system in terms of nominal interest rate and aggregate consumption C. Second, we combine the solution to the dynamic system with the Slutsky equation to obtain our consumption decomposition.

Solution to the dynamic system. The dynamic system is now given by

$$\begin{bmatrix}\dot{c}_t \\ \dot{\pi}_t\end{bmatrix} = \begin{bmatrix}-\delta & -\tilde{\sigma}^{-1} \\ -\kappa & \rho\end{bmatrix}\begin{bmatrix}c_t \\ \pi_t\end{bmatrix} + \begin{bmatrix}\nu_t \\ 0\end{bmatrix}$$

where $\nu_t \equiv \tilde{\sigma}^{-1}(i_t - \rho) + v_t$.

The eigenvalues of the system are given by

$$\overline{\omega}_T \equiv \frac{\rho - \delta + \sqrt{(\rho-\delta)^2 + 4(\rho\delta + \kappa\tilde{\sigma}^{-1})}}{2}, \qquad \underline{\omega}_T \equiv \frac{\rho - \delta - \sqrt{(\rho-\delta)^2 + 4(\rho\delta + \kappa\tilde{\sigma}^{-1})}}{2}$$

Suppose that $\hat{\sigma}^{-1} > 0$. Then, $\overline{\omega}_T > 0$ and $\underline{\omega}_T < 0$, given that $\delta \geq 0$. We can write the system in matrix form as follows:

$$\dot{Z}_t = AZ_t + B\nu_t$$

The matrix A can be written as

$$A = V\Omega V^{-1}$$

where

$$V = \begin{bmatrix} \frac{\rho-\overline{\omega}}{\kappa} & \frac{\rho-\underline{\omega}}{\kappa} \\ 1 & 1 \end{bmatrix}; \qquad V^{-1} = \frac{\kappa}{\overline{\omega}-\underline{\omega}} \begin{bmatrix} -1 & \frac{\rho-\underline{\omega}}{\kappa} \\ 1 & -\frac{\rho-\overline{\omega}}{\kappa} \end{bmatrix}; \qquad \Omega = \begin{bmatrix} \overline{\omega} & 0 \\ 0 & \underline{\omega} \end{bmatrix}$$

Decoupling the system, we obtain

$$\dot{z}_t = \Omega z_t + b\nu_t$$

where $z_t = V^{-1}Z_t$ and $b = V^{-1}B$.

Solving the equation with positive eigenvalue forward and the one with negative eigenvalue backward, we obtain

$$\begin{aligned} z_{1,t} &= -b_1 \int_t^\infty e^{-\overline{\omega}(z-t)}\nu_z dz \\ z_{2,t} &= e^{\underline{\omega}t} z_{2,0} + b_2 \int_0^t e^{\underline{\omega}(t-z)}\nu_z dz \end{aligned}$$

Consumption and inflation are then given by

$$\begin{aligned} c_t &= V_{12}\left(V^{21}c_0 + V^{22}\pi_0\right)e^{\underline{\omega}t} - V_{11}V^{11}\int_t^\infty e^{-\overline{\omega}(z-t)}\nu_z dz + V_{12}V^{21}\int_0^t e^{\underline{\omega}(t-z)}\nu_z dz \\ \pi_t &= V_{22}\left(V^{21}c_0 + V^{22}\pi_0\right)e^{\underline{\omega}t} - V_{21}V^{11}\int_t^\infty e^{-\overline{\omega}(z-t)}\nu_z dz + V_{22}V^{21}\int_0^t e^{\underline{\omega}(t-z)}\nu_z dz \end{aligned}$$

Average consumption is given by

$$C = V_{12}\left(V^{21}c_0 + V^{22}\pi_0\right)\frac{\rho}{\rho-\underline{\omega}} - \frac{\rho}{\rho-\overline{\omega}}V_{11}V^{11}\int_0^\infty \left(e^{-\overline{\omega}z} - e^{-\rho z}\right)\nu_z dz + \frac{\rho}{\rho-\underline{\omega}}V_{12}V^{21}\int_0^\infty e^{-\rho z}\nu_z dz$$

Consumption is then given by

$$c_t = \frac{\rho-\underline{\omega}}{\rho}Ce^{\underline{\omega}t} + c_t^*$$

where

$$c_t^* = -V_{11}V^{11}\int_t^\infty e^{-\overline{\omega}(z-t)}\nu_z dz + V_{12}V^{21}\int_0^t e^{\underline{\omega}(t-z)}\nu_z dz + \left(\frac{\rho-\underline{\omega}}{\rho-\overline{\omega}}V_{11}V^{11}\int_0^\infty \left(e^{-\overline{\omega}z} - e^{-\rho z}\right)\nu_z dz - V_{12}V^{21}\int_0^\infty e^{-\rho z}\nu_z dz\right)e^{\underline{\omega}t}.$$

Inflation is given by

$$\pi_t = \frac{\kappa}{\rho-\underline{\omega}}Ce^{\underline{\omega}t} + \pi_t^*,$$

where

$$\pi_t^* = -V_{21}V^{11}\int_t^\infty e^{-\overline{\omega}(z-t)}\nu_z dz + V_{22}V^{21}\int_0^t e^{\underline{\omega}(t-z)}\nu_z dz + \frac{\kappa}{\rho-\underline{\omega}}\left(\frac{\rho-\underline{\omega}}{\rho-\overline{\omega}}V_{11}V^{11}\int_0^\infty \left(e^{-\overline{\omega}z} - e^{-\rho z}\right)\nu_z dz - V_{12}V^{21}\int_0^\infty e^{-\rho z}\nu_z dz\right)e^{\underline{\omega}t}.$$

Note that we can write c_t^* and π_t^* as follows:

$$c_t^* = -\frac{(\overline{\omega}-\rho)e^{\overline{\omega}t} - (\underline{\omega}-\rho)e^{\underline{\omega}t}}{\overline{\omega}-\underline{\omega}}\int_t^\infty e^{-\overline{\omega}z}\nu_z dz + \frac{\rho-\underline{\omega}}{\overline{\omega}-\underline{\omega}}\int_0^t \left(e^{-\underline{\omega}z} - e^{-\overline{\omega}z}\right)e^{\underline{\omega}t}\nu_z dz$$

$$\pi_t^* = \frac{\kappa}{\overline{\omega}-\underline{\omega}}\left(e^{\overline{\omega}t} - e^{\underline{\omega}t}\right)\int_t^\infty e^{-\overline{\omega}z}\nu_z dz + \frac{\kappa}{\overline{\omega}-\underline{\omega}}e^{\underline{\omega}t}\int_0^t \left(e^{-\underline{\omega}z} - e^{-\overline{\omega}z}\right)\nu_z dz.$$

where

$$\nu_t = \frac{1-\omega}{1-\omega\chi_c}\sigma^{-1}(i_t-\rho) - \frac{\omega\chi_r}{1-\omega\chi_c}\dot{i}_t.$$

Consumption Decomposition. Consumption of borrowers can be written as

$$\begin{aligned} c_{b,t} &= \chi_c\left(\omega c_{b,t} + (1-\omega)c_{s,t}\right) - \chi_r(i_t - \pi_t - \rho) \\ &= \frac{1-\omega}{1-\omega\chi_c}\chi_c c_{s,t} - \frac{\chi_r}{1-\omega\chi_c}(i_t - \pi_t - \rho) \end{aligned}$$

Consumption of savers can be decomposed using the Slutsky decomposition

$$c_{s,t} = c^H_{s,t} + C_s$$

where

$$C_s = \frac{1-\omega\chi_c}{1-\omega}C + \frac{\omega\chi_r}{1-\omega}\rho\int_0^\infty e^{-\rho t}(i_t - \pi_t - \rho)dt$$

Aggregate consumption can then be written as

$$\begin{aligned} c_t &= \omega\left(\frac{1-\omega}{1-\omega\chi_c}\chi_c\left(c^H_{s,t} + C_s\right) - \frac{\chi_r}{1-\omega\chi_c}(i_t - \pi_t - \rho)\right) + (1-\omega)\left(c^H_{s,t} + C_s\right) \\ &= \omega\left(\frac{1-\omega}{1-\omega\chi_c}\chi_c c^H_{s,t} - \frac{\chi_r}{1-\omega\chi_c}(i_t - \pi_t - \hat{r})\right) + C + (1-\omega)c^H_{s,t} \end{aligned}$$

where $\hat{r} = \rho\int_0^\infty e^{-\rho t}(i_t - \pi_t)$.

Using the expression (A) for π_t, we obtain

$$c_t = (1-\omega)c^{S,*}_t + \omega\frac{1-\omega}{1-\omega\chi_c}\chi_c c^{S,*}_t - \frac{\omega\chi_r}{1-\omega\chi_r}(i_t - \pi^*_t - \hat{r}^*) + C + \left(\frac{\overline{\omega}+\delta}{\rho}e^{\underline{\omega}t} - 1\right)C$$

where $\hat{r}^* \equiv \rho\int_0^\infty e^{-\rho s}(i_s - \pi^*_s)ds$ and $c^{S,*}_t$ is the Hicksian demand of savers evaluated at π^*_t.

■

B Hicksian Demand

B.1 Derivation of the Hicksian demand

The Hicksian demand of the non-linear model is obtained as the solution to the following problem

$$\min_{\{C_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\int_0^t (i_s-\pi_s)ds} C_t dt$$

$$st \quad \int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\sigma}}{1-\sigma} dt \geq \overline{U},$$

for some $\overline{U} \in \mathbb{R}$. The FOCs of this problem are given by

$$e^{-\int_0^t (i_s-\pi_s)ds} = \lambda e^{-\rho t} C_t^{-\sigma} \quad \forall t,$$

where λ is the Lagrange multiplier associated to the constraint. This implies that

$$C_t = e^{\frac{1}{\sigma}\int_0^t (i_s-\pi_s)ds} \lambda^{\frac{1}{\sigma}} e^{-\frac{\rho}{\sigma}t} \implies e^{-\rho t} C_t^{1-\sigma} = e^{-\frac{\rho}{\sigma}t} e^{-\frac{\sigma-1}{\sigma}\int_0^t (i_s-\pi_s)ds} \lambda^{\frac{1-\sigma}{\sigma}},$$

and hence

$$\lambda = \left[\frac{(1-\sigma)\overline{U}}{\int_0^{\infty} e^{-\frac{\rho}{\sigma}t} e^{-\frac{\sigma-1}{\sigma}\int_0^t (i_s-\pi_s)ds} dt}\right]^{\frac{\sigma}{1-\sigma}}.$$

Replacing in the FOC for C_t, we get

$$C_t = \frac{e^{-\frac{\rho}{\sigma}t} e^{\frac{1}{\sigma}\int_0^t (i_s-\pi_s)ds}}{\left[\int_0^{\infty} e^{-\frac{\rho}{\sigma}t} e^{-\frac{\sigma-1}{\sigma}\int_0^t (i_s-\pi_s)ds} dt\right]^{\frac{1}{1-\sigma}}} \left[(1-\sigma)\overline{U}\right]^{\frac{1}{1-\sigma}}.$$

Log-linearizing around the zero inflation steady state we get,

$$c_t^S = \frac{1}{\sigma}\int_0^t (i_s-\pi_s-\rho)ds - \frac{1}{\sigma}\int_0^{\infty} e^{-\rho t}(i_s-\pi_s-\rho)dt.$$

The present discounted value of the substitution effect is given by

$$\begin{aligned}
\int_0^{\infty} e^{-\rho t} c_t^S dt &= \frac{1}{\sigma}\int_0^{\infty} e^{-\rho t}\int_0^t (i_s-\pi_s-\rho)dsdt - \frac{1}{\sigma}\int_0^{\infty} e^{-\rho t}\int_0^{\infty} e^{-\rho s}(i_s-\pi_s-\rho)ds, \\
&= \frac{1}{\rho\sigma}\int_0^{\infty} e^{-\rho t}(i_s-\pi_s-\rho)ds - \frac{1}{\rho\sigma}\int_0^{\infty} e^{-\rho s}(i_s-\pi_s-\rho)ds, \\
&= 0.
\end{aligned}$$

B.2 The Neo-Fisherian property of the Hicksian inflation

As derived in the proof of lemma 3, the Hicksian inflation is given by

$$\pi_t^S = \frac{\sigma^{-1}\kappa}{\overline{\omega} - \underline{\omega}} e^{\underline{\omega} t} \int_0^t \left(e^{-\underline{\omega} s} - e^{-\overline{\omega} s}\right)(i_s - \rho)\, ds + \frac{\sigma^{-1}\kappa}{\overline{\omega} - \underline{\omega}} \left(e^{\overline{\omega} t} - e^{\underline{\omega} t}\right) \int_t^\infty e^{-\overline{\omega} s} (i_s - \rho)\, ds$$

The derivative of π_t^S with respect to i_s is given by

$$\frac{\partial \pi_t^S}{\partial i_s} = \begin{cases} \frac{\sigma^{-1}\kappa}{\overline{\omega}-\underline{\omega}} e^{\underline{\omega} t} \left(e^{-\underline{\omega} s} - e^{-\overline{\omega} s}\right) > 0, & \text{if } t > s \\ \frac{\sigma^{-1}\kappa}{\overline{\omega}-\underline{\omega}} (e^{\overline{\omega} t} - e^{\underline{\omega} t}) e^{-\overline{\omega} s} > 0, & \text{if } t \leq s \end{cases}$$

for $t > 0$.

Hence, Hicksian inflation reacts positively to past or future nominal interest rate, for $t > 0$.

C Derivation of TANK model in continuous time

Environment

Households. We consider an economy with two types of households that differ in their discount rates. There is a mass ω of high-discount agents denoted by *borrowers* and indexed by b, and a mass $1-\omega$ of low-discount agents denoted by *savers* and indexed by s. [43]

Households receive labor income $W_t N_{j,t}$, profits from corporate holdings $\Pi_{j,t}$, and government transfers $P_t \tilde{T}_{j,t}$, for $j \in \{b, s\}$. We assume that corporations are owned by savers, so $\Pi_{b,t} = 0$ for $t \geq 0$. The discount rate of household j is denoted by ρ_j, where $\rho_b \geq \rho_s$. Households are subject to a borrowing constraint that limits the maximum amount of debt they can have.

The problem of household $j \in \{b, s\}$ is given by

$$\max_{\{C_{j,t}, N_{k,t}\}_0^\infty} \int_0^\infty e^{-\rho_j t} \left[\frac{C_{j,t}^{1-\sigma}}{1-\sigma} - \frac{N_{j,t}^{1+\phi}}{1+\phi} \right] dt,$$

subject to the flow budget constraint

$$\dot{B}_{j,t} = i_t B_{j,t} + W_t N_{j,t} + \Pi_{j,t} + P_t \tilde{T}_{j,t} - P_t C_{j,t},$$

and the borrowing constraint

$$\frac{B_{j,t}}{P_t} \geq -\overline{D}_p,$$

where $\overline{D}_p$ denotes the maximum amount of private debt.

The optimality conditions for this problem can be written as

$$\frac{W_t}{P_t} = N_{j,t}^{\phi} C_{j,t}^{\sigma}$$
$$\frac{\dot{C}_{j,t}}{C_{j,t}} = \frac{i_t - \pi_t - \rho_j}{\sigma} + \mu_{j,t}$$

where $\mu_{j,t} > 0$ if the borrowing constraint is binding and $\mu_{j,t} = 0$ if it is slack.

Firms. There are two types of firms in the economy: final-goods producers and intermediate-goods producers. Final-goods producers operate in a perfectly competitive market and combine a unit mass of intermediate goods $Y_t(j)$, for $j \in [0, 1]$, using the production func-

[43] Note that the RANK model is a special case in which $\omega = 0$.

tion

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}} \tag{32}$$

The problem of the final-good producer is given by

$$\max_{[Y_t(j)]_{j\in[0,1]}} P_t Y_t - \int_0^1 P_t(j) Y_t(j) dj$$

subject to (32).

The solution to the problem above gives the standard CES demand

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t$$

where $P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}$.

Intermediate goods are produced using the production function $Y_t(j) = N_t(j)$. Sales of intermediate goods are taxed at the constant rate τ. Firms choose prices subject to Calvo pricing, that is, they choose $P_t^*(j)$ to maximize

$$\max_{P_t^*(j)} \int_t^\infty \rho_\delta e^{-\int_t^z (i_s+\rho_\delta)ds} \left[(1-\tau)P_t^*(j)\left(\frac{P_t(j)^*}{P_z}\right)^{-\epsilon} Y_z - W_z \left(\frac{P_t(j)^*}{P_z}\right)^{-\epsilon} Y_z\right] dz$$

where ρ_δ is the arrival rate of the Poisson process determining the periods of price adjustment.

The first-order condition is given by

$$P_t^*(j) = \frac{\epsilon}{\epsilon-1}\int_t^\infty \frac{e^{-\int_t^z (i_s+\rho_\delta)ds} P_z^\epsilon Y_z(j)}{\int_t^\infty e^{-\int_t^{z'} (i_s+\rho_\delta)ds} P_{z'}^\epsilon Y_{z'}(j) dz'} \frac{W_z}{1-\tau} dz$$

Government. The government chooses transfers to borrowers and savers, $\{\tilde{T}_{b,t}, \tilde{T}_{s,t}\}_{t=0}^\infty$, and the sales tax rate τ to satisfy the flow budget constraint

$$\dot{D}_{g,t} = i_t D_{g,t} + P_t\left(\omega \tilde{T}_{b,t} + (1-\omega)\tilde{T}_{s,t}\right) - \tau \int_0^1 P_t(j) Y_t(j) dj$$

and the No-Ponzi condition $\lim_{t\to} e^{-\int_0^t i_s ds} D_{g,t} \leq 0$.

Combining the conditions above, we obtain the intertemporal budget constraint of the government

$$D_{g,0} = \int_0^\infty e^{-\int_0^t i_s ds} P_t\left[\tau Y_t - \left(\omega \tilde{T}_{b,t} + (1-\omega)\tilde{T}_{s,t}\right)\right] dt$$

Market clearing conditions. The market clearing conditions for goods, labor, and bonds are given by

$$\omega C_{b,t} + (1-\omega)C_{s,t} = Y_t$$
$$\omega N_{b,t} + (1-\omega)N_{s,t} = N_t$$
$$\omega B_{b,t} + (1-\omega)B_{s,t} = D_{g,t}$$

where $N_t = \int_0^1 N_t(j)dj$ denotes the aggregate labor demand in period t.

Steady state

In steady state, we have that $\mu_{j,t} = \sigma^{-1}(\rho_j - r)$, where r is the steady-state real interest rate. Therefore, $r = \rho_s$ and $\mu_{b,t} > 0$ for all t, so borrowers are against the borrowing constraint. To ease notation, we write $\rho_s = \rho$.

Consumption of borrowers is given by

$$C_b = \frac{W}{P}N_b + \tilde{T}_b - \rho\overline{D}_p$$

Consumption of savers is given by

$$C_s = \frac{W}{P}N_s + \frac{(1-\tau)Y - \frac{W}{P}N}{1-\omega} + \tilde{T}_s + \rho B_s$$

Note that by combining the two conditions above and market clearing, we obtain the government's budget constraint

$$\tau Y - \omega\tilde{T}_b - (1-\omega)\tilde{T}_s = \rho\overline{D}_g$$

where $\overline{D}_g \geq 0$ is the amount of government debt in steady state.

From the optimal pricing condition, we obtain

$$P = \frac{\epsilon}{\epsilon - 1}\frac{W}{1-\tau}$$

The distribution of consumption in steady state will depend on fiscal policy. Fix a steady state with a given value for (C_b, C_s) and government debt $\overline{D}_g$. The required value of transfers that implement the given level of consumption are

$$\tilde{T}_b = C_b - \left(\frac{W}{P}\right)^{\frac{1+\phi}{\phi}} C_b^{-\frac{\sigma}{\phi}} + \rho\overline{D}_p$$

$$\tilde{T}_s = C_s - \left(\frac{W}{P}\right)^{\frac{1+\phi}{\phi}} C_s^{-\frac{\sigma}{\phi}} - \frac{1}{\epsilon}\frac{1-\tau}{1-\omega}Y - \rho\frac{\overline{D}_g + \omega\overline{D}_p}{1-\omega}$$

where $Y = \omega C_b + (1-\omega)C_s$, $\frac{W}{P} = (1-\tau)(1-\epsilon^{-1})$, and we used the labor supply conditions to eliminate N_b and N_s.

Equilibrium dynamics

Exponentially decaying interest rates. Suppose $i_t - \rho = e^{-\theta t}(i_0 - \rho)$. Then, ν_t is given by

$$\nu_t = \hat{\sigma}^{-1}(i_t - \rho).$$

where $\hat{\sigma}^{-1} \equiv \frac{(1-\omega)\sigma^{-1} + \omega\chi_r\theta}{1-\omega\chi_c}$.

The consumption path evaluated at $C = 0$, c_t^*, is given by

$$c_t^* = -\frac{(\overline{\omega}-\rho) - (\underline{\omega}-\rho)e^{-(\overline{\omega}-\underline{\omega})t}}{\overline{\omega}-\underline{\omega}}\hat{\sigma}^{-1}\frac{i_t - \rho}{\overline{\omega}+\theta} + \frac{\rho - \underline{\omega}}{\overline{\omega}-\underline{\omega}}\left(\frac{1-e^{-(\underline{\omega}+\theta)t}}{\underline{\omega}+\theta} - \frac{1-e^{-(\overline{\omega}+\theta)t}}{\overline{\omega}+\theta}\right)e^{\underline{\omega}t}\hat{\sigma}^{-1}(i_0 - \rho)$$

$$= \hat{\sigma}^{-1}\frac{(\rho-\underline{\omega})e^{\underline{\omega}t} - (\rho+\theta)e^{-\theta t}}{(\underline{\omega}+\theta)(\overline{\omega}+\theta)}(i_0 - \rho).$$

The inflation path evaluated at $C = 0$, π_t^*, is given by

$$\pi_t^* = \frac{\kappa}{\overline{\omega}-\underline{\omega}}\left(1 - e^{-(\overline{\omega}-\underline{\omega})t}\right)\frac{\hat{\sigma}^{-1}}{\overline{\omega}+\theta}(i_t - \rho) + \frac{\kappa}{\overline{\omega}-\underline{\omega}}\left(\frac{1-e^{-(\underline{\omega}+\theta)t}}{\underline{\omega}+\theta} - \frac{1-e^{-(\overline{\omega}+\theta)t}}{\overline{\omega}+\theta}\right)e^{\underline{\omega}t}\hat{\sigma}^{-1}(i_0 - \rho)$$

$$= \kappa\hat{\sigma}^{-1}\frac{e^{\underline{\omega}t} - e^{-\theta t}}{(\overline{\omega}+\theta)(\underline{\omega}+\theta)}(i_0 - \rho).$$

The real rate is given by

$$i_t - \pi_t^* - \rho = \left(1 + \frac{\kappa\hat{\sigma}^{-1}}{(\overline{\omega}+\theta)(\underline{\omega}+\theta)}\right)e^{-\theta t}(i_0 - \rho) - \frac{\kappa\hat{\sigma}^{-1}}{(\overline{\omega}+\theta)(\underline{\omega}+\theta)}e^{\underline{\omega}t}(i_0 - \rho),$$

and the average real rate is given by

$$\rho\int_0^\infty e^{-\rho t}(i_t - \pi_t^* - \rho)dt = \left(1 + \frac{\kappa\hat{\sigma}^{-1}}{(\overline{\omega}+\theta)(\underline{\omega}+\theta)}\right)\frac{\rho}{\rho+\theta}(i_0 - \rho) - \frac{\kappa\hat{\sigma}^{-1}}{(\overline{\omega}+\theta)(\underline{\omega}+\theta)}\frac{\rho}{\rho-\underline{\omega}}(i_0 - \rho),$$

The purely forward-looking solution to the dynamic system satisfies $C = \int_0^\infty \Omega_z dz$, where

$$\Omega_z = -\frac{\rho}{\overline{\omega} - \underline{\omega}} \left(e^{-\underline{\omega} z} - e^{-\overline{\omega} z} \right)$$

Hence, C can be written as

$$C = -\frac{\rho \hat{\sigma}^{-1}}{(\overline{\omega} + \theta)(\underline{\omega} + \theta)} (i_0 - \rho)$$

D Derivation of the Quantitative Model: RANK

Goods production

Output is produced by a representative, competitive firm using the following technology

$$Y_t = \left(\int_0^1 Y_{i,t}^{\frac{1}{\lambda_f}} di \right)^{\lambda_f}, \quad \lambda_f > 1,$$

where λ_f governs the degree of substitution between the different inputs. Profit maximization leads to the following FOC

$$Y_{i,t} = \left(\frac{P_{i,t}}{P_t} \right)^{-\frac{\lambda_f}{\lambda_f - 1}} Y_t.$$

Thus, the aggregate price level is

$$P_t = \left(\int_0^1 P_{i,t}^{\frac{1}{1-\lambda_f}} di \right)^{1-\lambda_f}.$$

The production function of intermediate goods is given by

$$Y_{i,t} = \begin{cases} (z_t H_{i,t})^{1-\alpha} K_{i,t}^{\alpha} - z_t^+ \varphi & \text{if } (z_t H_{i,t})^{1-\alpha} K_{i,t}^{\alpha} \geq z_t^+ \varphi \\ 0 & \text{otherwise.} \end{cases}$$

Here, z_t is a technology shock and φ denotes a fixed production cost. The object z_t^+ is defined as follows:

$$z_t^+ = \Psi_t^{\frac{\alpha}{1-\alpha}} z_t.$$

Along a nonstochastic steady-state growth path, Y_t / z_t^+ and $Y_{i,t} / z_t^+$ converge to a constant. We assume that

$$\frac{z_t}{z_{t-1}} = \mu_z$$
$$\frac{\Psi_t}{\Psi_{t-1}} = \mu_\psi$$

hence

$$\frac{z_t^+}{z_{t-1}^+} = \mu_\Psi^{\frac{\alpha}{1-\alpha}} \mu_z.$$

As is common in the literature, we assume that there is no entry or exit of intermediate good producers, and that firms make zero profits in steady state. This requires that the fixed cost grow at the same rate as the growth rate of output; this is why φ is multiplied by z_t^+ in the intermediaries' production function.

The production of intermediate goods uses capital services, $K_{i,t}$, and homogeneous labor services, $H_{i,t}$. Firms must borrow to pay the wage bill, so that the cost of labor is given by

$$W_t R_t,$$

where W_t denotes the aggregate nominal wage rate and R_t denotes the gross nominal interest rate on working capital loans. In an interior solution, the optimal choice of capital and labor satisfies

$$\frac{r_t^k}{\frac{W_t R_t}{P_t}} = \frac{\alpha}{1-\alpha}\frac{H_{i,t}}{K_{i,t}}$$

and the firms' real marginal cost is given by

$$s_t = \frac{1}{z_t^{1-\alpha}}\left(\frac{r_t^k}{\alpha}\right)^{\alpha}\left(\frac{W_t R_t}{(1-\alpha)P_t}\right)^{1-\alpha}.$$

Apart from the fixed costs, the firms' time t real profits are

$$\left[(1-\tau_t)\frac{P_{i,t}}{P_t} - s_t\right] Y_{i,t}$$

where τ_t is a proportional sales tax.

We adopt the Calvo model of price frictions. With probability ξ_p the intermediate good firm cannot reoptimize its price, in which case it is assumed to set its price according to the following rule

$$P_{i,t} = \pi_{t-1} P_{i,t-1}.$$

With probability $1-\xi_p$ the intermediate good firm can reoptimize its price.

Let $\tilde{P}_t$ denote the value of $P_{i,t}$ set by a firm that can reoptimize at time t. Note that $\tilde{P}_t$ does not depend on i since all firms that can reoptimize their price at time t choose the same price. The firm chooses $\tilde{P}_t$ to maximize

$$E_{t-1}\sum_{j=0}^{\infty}(\beta\xi_p)^j v_{t+j} P_{t+j}\left[(1-\tau_{t+j})\frac{X_{j,t}^p \tilde{P}_t}{P_{t+j}} - s_{t+j}\right] Y_{i,t+j},$$

where

$$X_{j,t}^{p} = \begin{cases} \pi_t \times \pi_{t+1} \times \cdots \times \pi_{t+j-1} & \text{for } j \geq 1 \\ 1 & \text{for } j = 0, \end{cases}$$

and $\beta^j v_{t+j}$ is the household's valuation in period t of one unit of the final good delivered in period $t+j$. The first-order necessary condition associated with this optimization problem is given by

$$E_{t-1} \sum_{j=0}^{\infty} (\beta \xi_p)^j v_{t+j} P_{t+j} Y_{t+j} \left[(1 - \tau_{t+j}) \frac{X_{j,t}^{p} \tilde{P}_t}{P_{t+j}} - \lambda_f s_{t+j} \right] = 0$$

The goods market clearing condition for this economy is given by

$$Y_t = C_t + \tilde{I}_t + G_t,$$

where C_t denotes household consumption, G_t denotes exogenous government consumption, and $\tilde{I}_t$ is a homogeneous investment good that is defined as follows:

$$\tilde{I}_t = \frac{1}{\Psi_t} (I_t + a(u_t) \overline{K}_t).$$

The investment goods, I_t, are used by households to add to the physical stock of capital, $\overline{K}_t$. The remaining investment goods are used to cover maintenance costs, $a(u_t)\overline{K}_t$, arising from capital utilization, u_t. The cost function $a(\cdot)$ is increasing and convex, and has the property that in steady state, $u_t = 1$ and $a(1) = 0$. Finally, Ψ_t is the productivity of the production of the homogeneous investment good $\tilde{I}_t$.

The relationship between the utilization of capital, u_t, capital services, K_t, and the physical stock of capital, $\overline{K}_t$, is as follows

$$K_t = u_t \overline{K}_t.$$

The investment and capital utilization are discussed in the households' problem below.

Households

Households supply the factors of production, labor, and capital. The model incorporates Calvo-style wage setting frictions. We assume that there are many different specialized labor inputs, $h_{j,t}$, for $j \in (0,1)$. There is a single monopolist that sets the wage for each type, j, of labor service. However, the monopolist's market power is limited by the presence of other labor services, $j' \neq j$, that are substitutable for $h_{j,t}$. We assume that labor is indivisible:

people work either full time or not at all. That is, $h_{j,t}$ represents a quantity of people and not, say, the number of hours worked by a representative worker.

Households and the labor market

The labor hired by firms is interpreted as a homogeneous factor of production, H_t, supplied by "labor contractors." Labor contractors produce H_t by combining a range of differentiated labor inputs, $h_{j,t}$, using the following linear homogeneous technology:

$$H_t = \left[\int_0^1 (h_{j,t})^{\frac{1}{\lambda_w}} dj \right]^{\lambda_w}, \quad \lambda_w > 1.$$

Labor contractors are perfectly competitive and take the wage rate of H_t, W_t, as given. They also take the wage rate of the j^{th} labor type, $W_{j,t}$, as given. Contractors choose inputs and outputs to maximize profits

$$W_t H_t - \int_0^1 W_{j,t} h_{j,t} dj.$$

The first-order necessary condition for optimization is given by

$$h_{j,t} = \left(\frac{W_t}{W_{j,t}} \right)^{\frac{\lambda_w}{\lambda_w - 1}} H_t. \tag{33}$$

Substituting the latter back into the labor aggregator function and rearranging, we obtain

$$W_t = \left(\int_0^1 W_{j,t}^{\frac{1}{1-\lambda_w}} dj \right)^{1-\lambda_w}.$$

Differentiated labor is supplied by a large number of identical households. The representative household has many members corresponding to each type, j, of labor. Each worker of type j has an index, l, distributed uniformly over the unit interval, $[0,1]$, which indicates that worker's disutility to work. A type j worker with index l experiences utility

$$\log(c_t^e - bC_{t-1}) - A_L l^{\phi}, \quad \phi > 0,$$

if employed, and

$$\log(c_t^{ne} - bC_{t-1}),$$

if not employed. When $b > 0$, the worker's marginal utility of current consumption is an increasing function of the household's consumption in the previous period. Given the additive separability of consumption and employment in utility, the efficient allocation of consumption across workers within the household implies

$$c_t^e = c_t^{ne} = C_t.$$

We assume that the household sends j-type workers with $0 \leq l \leq h_{j,t}$ to work and keeps those $l > h_{j,t}$ out of the labor force. The equally weighted integral of utility over all $l \in [0, 1]$ workers is

$$\log(C_t - bC_{t-1}) - A_L \frac{h_{j,t}^{1+\phi}}{1+\phi}.$$

Aggregate household utility also integrates over the unit measure of j-type workers

$$\log(C_t - bC_{t-1}) - A_L \int_0^1 \frac{h_{j,t}^{1+\phi}}{1+\phi} dj. \tag{34}$$

The wage rate of the j^{th} type of labor, $W_{j,t}$, is determined outside the representative household by a monopoly union that represents all j-type workers across all households. The union's problem is discussed below.

Wages, employment, and monopoly unions

In each period, the monopoly union must satisfy its demand curve, (33), and it faces Calvo frictions in the setting of $W_{j,t}$. With probability $1 - \xi_w$ the union can optimize the wage, and with the complementary probability, ξ_w, it cannot. In the latter case, we suppose that the nominal wage rate is set as follows:

$$W_{j,t+1} = \tilde{\pi}_{w,t} W_{j,t}$$
$$\tilde{\pi}_{w,t} = \pi_t \mu_{z^+}.$$

With this specification, the wage of each type j labor is the same in the steady state. Because the union problem has no state variable, all unions with the opportunity to reoptimize in the current period choose the same wage. In particular, a union that can reoptimize chooses the current value of the wage, $\tilde{W}_t$, to maximize

$$E_{t-1} \sum_{j=0}^{\infty} (\beta \xi_w)^j \left[v_{t+j} X_{j,t}^w \tilde{W}_t^t h_{i,t+j}^t - A_L \frac{(h_{i,t+j}^t)^{1+\phi}}{1+\phi} \right]$$

where

$$X_{j,t}^{w} = \begin{cases} \tilde{\pi}_{w,t} \times \tilde{\pi}_{w,t+1} \times \cdots \times \tilde{\pi}_{w,t+j-1} & \text{for } j \geq 1 \\ 1 & \text{for } j = 0. \end{cases}$$

Here, h_{t+j}^{t} denotes the quantity of workers employed in period $t+j$ of a union that has an opportunity to reoptimize the wage in period t and does not reoptimize again in periods $t+1, \ldots, t+j$. Also, v_{t+j} denotes the marginal value assigned by the representative household to the wage. The union treats v_t as an exogenous variable. The first order condition of the union's problem is

$$E_{t-1} \sum_{j=0}^{\infty} (\beta \xi_w)^j \left[v_{t+j} (X_{j,t}^{w})^{-\frac{1}{\lambda_w - 1}} W_{t+j}^{\frac{\lambda_w}{\lambda_w - 1}} \left(\tilde{W}_t^t\right)^{-\frac{1}{\lambda_w - 1}} H_{t+j} - \right.$$
$$\left. \lambda_w A_L (X_{j,t}^{w})^{-\frac{(1+\phi)\lambda_w}{\lambda_w - 1}} W_{t+j}^{\frac{(1+\phi)\lambda_w}{\lambda_w - 1}} \left(\tilde{W}_t^t\right)^{-\frac{(1+\phi)\lambda_w}{\lambda_w - 1}} H_{t+j}^{1+\phi} \right] = 0$$

Capital accumulation

The household owns the economy's physical stock of capital, sets the utilization rate of capital, and rents out the services of capital in a competitive market. The household accumulates capital using the following technology:

$$\overline{K}_{t+1} = (1 - \delta) \overline{K}_t + F\left(I_t, I_{t-1}\right), \tag{35}$$

where $\delta \in [0, 1]$ and

$$F\left(I_t, I_{t-1}\right) = \left(1 - S\left(\frac{I_t}{I_{t-1}}\right)\right) I_t.$$

We assume that $S\left(\mu_{z^+}\mu_{\Psi}\right) = S'\left(\mu_{z^+}\mu_{\Psi}\right) = 0$, where $\mu_{z^+}\mu_{\Psi}$ is the growth rate of investment in steady state. Also, we assume that $S''(\cdot) > 0$. Because of the nature of the above adjustment cost function, the curvature parameters have no impact on the model's steady state.

For each unit of $\overline{K}_{t+1}$ owned in period t, the household receives X_{t+1}^{k} in net nominal payments in period $t+1$

$$X_{t+1}^{k} = u_{t+1} P_{t+1} r_{t+1}^{k} - \frac{P_{t+1}}{\Psi_{t+1}} a\left(u_{t+1}\right). \tag{36}$$

The first term is the gross nominal period $t+1$ rental income from a unit of $\overline{K}_{t+1}$. The second term represents the cost of capital utilization, $a\left(u_{t+1}\right) P_{t+1} / \Psi_{t+1}$. Here, P_{t+1} / Ψ_{t+1} is the nominal price of the investment goods absorbed by capital utilization.

Household optimization problem

The household's period t budget constraint is as follows:

$$P_t\left(C_t+\frac{1}{\Psi_t}I_t\right)+B_t\leq\underbrace{\int_0^1 W_{t,j}h_{t,j}dj}_{=W_tH_t}+X_t^k\overline{K}_t+R_{t-1}B_{t-1}+P_tD_t+P_tT_t \tag{37}$$

where $W_{t,j}$ represents the wage earned by workers type-j; B_t denotes the quantity of one-period risk-free bonds purchased in period t, and R_{t-1} denotes the gross nominal interest rate on bonds purchased in period $t-1$, which pay off in period t; D_t are the firms' profits; and T_t are government lump-sum transfers. The household's problem is to select sequences, $\left\{C_t, I_t, B_{t+1}, \overline{K}_{t+1}\right\}$, to maximize (34) subject to the wage process selected by the monopoly unions, (35), (36), and (37).

The first-order necessary and sufficient conditions are

$$\begin{aligned}
(C_t):\quad & E_{t-1}\left[\frac{1}{C_t-bC_{t-1}}-\beta b\frac{1}{C_{t+1}-bC_t}\right]=v_tP_t\\
(u_t):\quad & E_{t-1}r_t^k=\frac{1}{\Psi_t}a'(u_t)\\
(\overline{K}_{t+1}):\quad & E_{t-1}P_tv_t=\beta E_{t-1}P_{t+1}v_{t+1}\left[\frac{u_{t+1}r_{t+1}^k-\frac{a(u_{t+1})}{\Psi_{t+1}}+(1-\delta)\mu_{t+1}}{\mu_t}\right]\\
(I_t):\quad & E_{t-1}P_t\frac{v_t}{\Psi_t}=E_{t-1}\left[P_tv_t\mu_tF_{1,t}+\beta P_{t+1}v_{t+1}\mu_{t+1}F_{2,t+1}\right]\\
(B_{t+1}):\quad & E_{t-1}v_t=E_{t-1}\beta v_{t+1}R_t
\end{aligned}$$

where $\beta^t v_t$ is the Lagrange multiplier associated with the budget constraint and $\beta^t\mu_t v_t$ is the Lagrange multiplier associated with the law of motion of capital.

Fiscal and monetary authorities

We suppose that monetary policy follows a Taylor rule of the following form:

$$\log\left(\frac{R_t}{R}\right)=\rho_R\log\left(\frac{R_{t-1}}{R}\right)+(1-\rho_R)\left[r_\pi\log\left(\frac{\pi_t}{\pi}\right)+r_y\log\left(\frac{Y_t}{z_t^+Y}\right)+\right.$$
$$\left.r_{\Delta y}\log\left(\frac{Y_t}{\mu_{z^+}Y_{t-1}}\right)\right]+\varepsilon_{R,t}$$

where $\varepsilon_{R,t}$ denotes a monetary policy shock, which follows

$$\log \varepsilon_{R,t} = \rho_M \log \varepsilon_{R,t-1} + u_t.$$

The per-period budget constraint of the government is given by

$$B_t = R_{t-1}B_{t-1} - \tau_t P_t Y_t + P_t T_t + P_t G_t.$$

We adopt the model of government consumption suggested in Christiano and Eichenbaum (1992 a):

$$G_t = g z_t^+.$$

Equilibrium and Steady State

An equilibrium is a stochastic process for the prices and quantities with the property that the household and firm problems are satisfied, and goods and labor markets clear. The equilibrium of this economy can be characterized by the following system of equations:

$$Y_{i,t} = (z_t H_{i,t})^{1-\alpha}(u_t \overline{K}_t)^{\alpha} - z_t^+ \varphi$$

$$\frac{r_t^k}{\frac{W_t R_t}{P_t}} = \frac{\alpha}{1-\alpha}\frac{H_{i,t}}{K_{i,t}}$$

$$s_t = \frac{1}{z_t^{1-\alpha}}\left(\frac{r_t^k}{\alpha}\right)^{\alpha}\left(\frac{W_t R_t}{P_t(1-\alpha)}\right)^{1-\alpha}$$

$$E_{t-1}\sum_{j=0}^{\infty}(\beta\xi_p)^j v_{t+j}P_{t+j}Y_{t+j}\left[(1-\tau_{t+j})\frac{X_{j,t}^p \tilde{P}_t}{P_{t+j}} - \lambda_f s_{t+j}\right] = 0$$

$$P_t = \left[(1-\xi_p)\tilde{P}_t^{\frac{1}{1-\lambda_f}} + \xi_p(\pi_{t-1}P_{t-1})^{\frac{1}{1-\lambda_f}}\right]^{1-\lambda_f}$$

$$Y_t = \left(\int_0^1 Y_{i,t}^{\frac{1}{\lambda_f}}\, di\right)^{\lambda_f}$$

$$Y_t = C_t + \frac{1}{\Psi_t}(I_t + a(u_t)\overline{K}_t) + G_t$$

$$E_{t-1}\sum_{j=0}^{\infty}(\beta\xi_w)^j\left[v_{t+j}(X^w_{j,t})^{-\frac{1}{\lambda_w-1}}W_{t+j}^{\frac{\lambda_w}{\lambda_w-1}}\left(\tilde{W}^t_t\right)^{-\frac{1}{\lambda_w-1}}H_{t+j}-\right.$$

$$\left.\lambda_w A_L(X^w_{j,t})^{-\frac{(1+\phi)\lambda_w}{\lambda_w-1}}W_{t+j}^{\frac{(1+\phi)\lambda_w}{\lambda_w-1}}\left(\tilde{W}^t_t\right)^{-\frac{(1+\phi)\lambda_w}{\lambda_w-1}}H_{t+j}^{1+\phi}\right]=0$$

$$W_t=\left[(1-\xi_w)\tilde{W}_t^{\frac{1}{1-\lambda_w}}+\xi_w(\pi_{t-1}\mu_{z^+}W_{t-1})^{\frac{1}{1-\lambda_w}}\right]^{1-\lambda_w}$$

$$\overline{K}_{t+1}=(1-\delta)\overline{K}_t+\left[1-S\left(\frac{I_t}{I_{t-1}}\right)\right]I_t$$

$$P_t\left(C_t+\frac{1}{\Psi_t}I_t\right)+B_t\leq W_tH_t+X^k_t\overline{K}_t+R_{t-1}B_{t-1}+P_tD_t+P_tT_t$$

$$\frac{1}{C_t-bC_{t-1}}-\beta bE_{t-1}\left[\frac{1}{C_{t+1}-bC_t}\right]=v_tP_t$$

$$r^k_t=\frac{1}{\Psi_t}a'(u_t)$$

$$P_tv_t=\beta E_{t-1}P_{t+1}v_{t+1}\left[\frac{u_{t+1}r^k_{t+1}-\frac{a(u_{t+1})}{\Psi_{t+1}}+(1-\delta)\mu_{t+1}}{\mu_t}\right]$$

$$P_t\frac{v_t}{\Psi_t}=E_{t-1}\left[P_tv_t\mu_t\left[1-S\left(\frac{I_t}{I_{t-1}}\right)-S'\left(\frac{I_t}{I_{t-1}}\right)\frac{I_t}{I_{t-1}}\right]+\beta P_{t+1}v_{t+1}\mu_{t+1}S'\left(\frac{I_{t+1}}{I_t}\right)\left(\frac{I_{t+1}}{I_t}\right)^2\right]$$

$$v_t=E_{t-1}\beta v_{t+1}R_t$$

$$\log\left(\frac{R_t}{R}\right)=\rho_R\log\left(\frac{R_{t-1}}{R}\right)+(1-\rho_R)\left[r_\pi\log\left(\frac{\pi_t}{\pi}\right)+r_y\log\left(\frac{Y_t}{z^+_tY}\right)+\right.$$

$$\left.r_{\Delta y}\log\left(\frac{Y_t}{\mu_{z^+}Y_{t-1}}\right)\right]+\varepsilon_{R,t}$$

$$\log\varepsilon_{R,t}=\rho_M\log\varepsilon_{R,t-1}+u_t$$

$$B_t=R_{t-1}B_{t-1}-\tau_tP_tY_t+P_tT_t+P_tG_t$$

Since the economy exhibits exogenous growth through z_t and Ψ_t, it is useful to rescale the variables in such a way that there exists a steady-state equilibrium in the rescaled variables. Let

$$y_{i,t}=\frac{Y_{i,t}}{z_t^+},\quad y_t=\frac{Y_t}{z_t^+},\quad c_t=\frac{C_t}{z_t^+},\quad i_t=\frac{I_t}{z_t^+\Psi_t},\quad \overline{k}_{t+1}=\frac{\overline{K}_{t+1}}{z_t^+\Psi_t},\quad b_t=\frac{B_t}{z_t^+P_t},\quad T_t=\frac{T_t}{z_t^+}$$
$$\pi_t=\frac{P_t}{P_{t-1}},\quad \tilde{p}_t=\frac{\tilde{P}_t}{P_t},\quad \tilde{w}_t=\frac{\tilde{W}_t^t}{W_t},\quad \overline{w}_t=\frac{W_t}{z_t^+P_t},\quad \overline{r}_t^k=\Psi_t r_t^k,\quad \psi_t=P_t v_t z_t^+,\quad \tilde{\mu}_t=\Psi_t\mu_t$$

Thus, we can rewrite the system characterizing equilibrium as

$$y_{i,t}=\frac{\mu_z}{\mu_{z^+}}H_{i,t}^{1-\alpha}(u_t\overline{k}_t)^{\alpha}-\varphi \tag{38}$$

$$\frac{\overline{r}_t^k}{\overline{w}_tR_t}=\mu_\Psi\mu_{z^+}\frac{\alpha}{1-\alpha}\frac{H_{i,t}}{K_{i,t}} \tag{39}$$

$$s_t=\left(\frac{\overline{r}_t^k}{\alpha}\right)^{\alpha}\left(\frac{\overline{w}_tR_t}{1-\alpha}\right)^{1-\alpha} \tag{40}$$

$$E_{t-1}\sum_{j=0}^{\infty}(\beta\xi_p)^j\psi_{t+j}y_{t+j}\left[(1-\tau_{t+j})\frac{\pi_t}{\pi_{t+j}}\tilde{p}_t-\lambda_f s_{t+j}\right]=0 \tag{41}$$

$$1=(1-\xi_p)\tilde{p}_t^{\frac{1}{1-\lambda_f}}+\xi_p\left(\frac{\pi_{t-1}}{\pi_t}\right)^{\frac{1}{1-\lambda_f}} \tag{42}$$

$$y_t=\left(\int_0^1 y_{i,t}^{\frac{1}{\lambda_f}}\,di\right)^{\lambda_f} \tag{43}$$

$$y_t=c_t+i_t+a(u_t)\frac{\overline{k}_t}{\mu_{z^+}\mu_\Psi}+g \tag{44}$$

$$E_{t-1}\sum_{j=0}^{\infty}(\beta\xi_w)^j\left[\psi_{t+j}\left(\frac{\prod_{s=0}^{j-1}\pi_{t+s}^{\kappa_w}\pi^{1-\kappa_w}}{\prod_{s=1}^{j}\pi_{t+s}}\right)^{-\frac{1}{\lambda_w-1}}\overline{w}_{t+j}^{\frac{\lambda_w}{\lambda_w-1}}\left(\overline{w}_t\tilde{w}_t^t\right)^{-\frac{1}{\lambda_w-1}}H_{t+j}-\right.$$
$$\left.\lambda_wA_L\left(\frac{\prod_{s=0}^{j-1}\pi_{t+s}^{\kappa_w}\pi^{1-\kappa_w}}{\prod_{s=1}^{j}\pi_{t+s}}\right)^{-\frac{(1+\phi)\lambda_w}{\lambda_w-1}}\overline{w}_{t+j}^{\frac{(1+\phi)\lambda_w}{\lambda_w-1}}\left(\overline{w}_t\tilde{w}_t^t\right)^{-\frac{(1+\phi)\lambda_w}{\lambda_w-1}}H_{t+j}^{1+\phi}\right]=0 \tag{45}$$

$$1 = (1-\xi_w) w_t^{\frac{1}{1-\lambda_w}} + \xi_w \left(\frac{\pi_{t-1}}{\pi_t} \frac{\overline{w}_{t-1}}{\overline{w}_t} \right)^{\frac{1}{1-\lambda_w}} \tag{46}$$

$$\bar{k}_{t+1} = \frac{1-\delta}{\mu_{z^+}\mu_\Psi} \bar{k}_t + \left[1 - S\left(\mu_{z^+}\mu_\Psi \frac{i_t}{i_{t-1}} \right) \right] i_t \tag{47}$$

$$c_t + i_t + b_t = (1-\overline{\tau}) y_t + \frac{R_{t-1}}{\pi_t \mu_z^+} b_{t-1} + T_t \tag{48}$$

$$\frac{1}{c_t - b\frac{c_{t-1}}{\mu_{z^+}}} - \beta b E_{t-1} \left[\frac{1}{\mu_{z^+} c_{t+1} - b c_t} \right] = \psi_t \tag{49}$$

$$\bar{r}_t^k = a'(u_t) \tag{50}$$

$$\mu_{z^+}\mu_\Psi \psi_t = \beta E_{t-1} \psi_{t+1} \left[\frac{u_{t+1}\bar{r}_{t+1}^k - a(u_{t+1}) + (1-\delta)\tilde{\mu}_{t+1}}{\tilde{\mu}_t} \right] \tag{51}$$

$$\begin{aligned} \frac{\psi_t}{\mu_{z^+}\mu_\Psi} = E_{t-1} \Bigg[\psi_t \tilde{\mu}_t \left[1 - S\left(\mu_{z^+}\mu_\Psi \frac{i_t}{i_{t-1}} \right) - S'\left(\mu_{z^+}\mu_\Psi \frac{i_t}{i_{t-1}} \right) \frac{i_t}{i_{t-1}} \right] + \\ \beta \psi_{t+1} \tilde{\mu}_{t+1} S'\left(\mu_{z^+}\mu_\Psi \frac{i_{t+1}}{i_t} \right) \left(\frac{i_{t+1}}{i_t} \right)^2 \Bigg] \end{aligned} \tag{52}$$

$$\psi_t = E_{t-1} \beta \psi_{t+1} \frac{R_t}{\mu_z^+ \pi_{t+1}} \tag{53}$$

$$\begin{aligned} \log\left(\frac{R_t}{R}\right) = \rho_R \log\left(\frac{R_{t-1}}{R}\right) + (1-\rho_R) \Bigg[r_\pi \log\left(\frac{\pi_t}{\pi}\right) + r_y \log\left(\frac{y_t}{Y}\right) + \\ r_{\Delta y} \log\left(\frac{y_t}{y_{t-1}}\right) \Bigg] + \varepsilon_{R,t} \end{aligned} \tag{54}$$

$$\varepsilon_{R,t} = \rho_M \varepsilon_{R,t-1} + u_t \tag{55}$$

$$b_t = \frac{R_{t-1}}{\pi_t \mu_{z^+}} b_{t-1} - \tau_t y_t + T_t + g \tag{56}$$

Finally, a steady state in the transformed variables is a solution to

$$y = \frac{\mu_z}{\mu_{z^+}} H^{1-\alpha} \bar{k}^{\alpha} - \varphi$$

$$\frac{\bar{r}^k}{\overline{w}R} = \mu_\Psi \mu_{z^+} \frac{\alpha}{1-\alpha} \frac{H}{K}$$

$$s = \left(\frac{\bar{r}^k}{\alpha} \right)^{\alpha} \left(\frac{\overline{w}R}{1-\alpha} \right)^{1-\alpha}$$

$$(1-\tau) - \lambda_f s = 0$$

$$y = c + i + g$$

$$\overline{w} = \frac{\lambda_w A_L H^{\phi}}{\psi}$$

$$i = \left[1 - \frac{1-\delta}{\mu_{z^+}\mu_\Psi} \right] \bar{k}$$

$$c + i + b = (1 - \overline{\tau}) + \frac{1}{\beta} b + T$$

$$\psi = \frac{\mu_{z^+} - \beta b}{c(\mu_{z^+} - b)}$$

$$\bar{r}^k = a'(1)$$

$$\mu_{z^+}\mu_\Psi = \beta \left[\frac{\bar{r}^k + (1-\delta)\tilde{\mu}}{\tilde{\mu}} \right]$$

$$\tilde{\mu} = \frac{1}{\mu_{z^+}\mu_\Psi}$$

$$\frac{\beta R}{\mu_{z^+}\pi} = 1$$

$$b = \frac{R}{\pi \mu_{z^+}} b - \tau y + T + g$$

Linearization

To solve the model numerically, we linearize the system (38)-(56) around a non-stochastic steady state in which the inflation rate is π. We define the following variables:

$$\hat{y}_t = \log \frac{y_t}{y}, \quad \hat{y}_{i,t} = \log \frac{y_{i,t}}{y}, \quad \hat{h}_t = \log \frac{h_t}{h}, \quad \hat{h}_{i,t} = \log \frac{h_{i,t}}{h}, \quad \hat{\bar{k}}_t = \log \frac{\bar{k}_t}{\bar{k}}, \quad \hat{\bar{k}}_{i,t} = \log \frac{\bar{k}_{i,t}}{\bar{k}}$$

$$\hat{u}_t = \log u_t, \quad \hat{\bar{r}}^k_t = \log \frac{\bar{r}^k_t}{\bar{r}^k}, \quad \hat{\bar{w}}_t = \log \frac{\overline{w}_t}{\overline{w}}, \quad \hat{s}_t = \log \frac{s_t}{s}, \quad \hat{\tilde{p}}_t = \log \tilde{p}_t, \quad \hat{\pi}_t = \log \frac{\pi_t}{\pi}$$

$$\hat{\tau} = \tau_t - \overline{\tau}, \quad \hat{c}_t = \log \frac{c_t}{c}, \quad \hat{i}_t = \log \frac{i_t}{i}, \quad \hat{\tilde{w}}_t = \log \tilde{w}_t, \quad \hat{\psi}_t = \log \frac{\psi_t}{\psi}, \quad \hat{\tilde{\mu}}_t = \log \frac{\tilde{\mu}_t}{\tilde{\mu}},$$

$$\hat{R}_t = \log \frac{R_t}{R}, \quad \hat{b}_t = \log \frac{b_t}{b}, \quad \hat{T}_t = \frac{T_t - \overline{T}}{y}.$$

Firms Block. Linearizing (38) around the non-stochastic steady state, we get

$$\hat{y}_t = \frac{y + \phi}{y} \left[(1-\alpha)\hat{h}_t + \alpha(\hat{u}_t + \hat{\bar{k}}_t) \right]. \tag{57}$$

where we used that $\int_0^1 \hat{y}_{i,t} = \hat{y}_t$, $\int_0^1 \hat{h}_{i,t} = \hat{h}_t$ and $\int_0^1 \hat{\bar{k}}_{i,t} = \hat{\bar{k}}_t$. Linearizing (39), we get

$$\hat{\bar{r}}^k_t - \hat{\bar{w}}_t - \hat{R}_t = \hat{h}_t - \hat{\bar{k}}_t$$

Combining this equation with (57), we get

$$\hat{\bar{w}}_t + \hat{R}_t + \frac{1}{1-\alpha}\frac{y}{y+\varphi}\hat{y}_t - \hat{\bar{r}}^k_t - \frac{\alpha}{1-\alpha}\hat{u}_t - \frac{1}{1-\alpha}\hat{k}_t = 0. \tag{58}$$

Linearizing (40), we get

$$\hat{s}_t = \alpha \hat{\bar{r}}^k_t + (1-\alpha)\left(\hat{\bar{w}}_t + \hat{R}_t\right) \tag{59}$$

Linearizing (41) around the non-stochastic steady state, we get

$$\frac{\hat{\tilde{p}}_t}{1-\beta\xi_p} = E_{t-1}\left[\hat{s}_t + \frac{\hat{\tau}_t}{1-\tau} + \frac{\beta\xi_p}{1-\beta\xi_p}(\hat{\tilde{p}}_{t+1} - \hat{\pi}_t + \hat{\pi}_{t+1})\right], \tag{60}$$

where, $\hat{\tau}_t = \tau_t - \overline{\tau}$, and the other hat variable indicates the percent deviation from its steady-state value. Linearizing (42) and rearranging, we obtain

$$\hat{\tilde{p}}_t = \frac{\xi_p}{1-\xi_p}(\hat{\pi}_t - \hat{\pi}_{t-1}). \tag{61}$$

Putting together (60) and (61), we get

$$\hat{\pi}_t = \frac{1}{1+\beta}\hat{\pi}_{t-1} + \frac{\beta}{1+\beta}E_{t-1}\hat{\pi}_{t+1} + \frac{\gamma}{1+\beta}E_{t-1}\left[\hat{s}_t + \frac{\hat{\tau}_t}{1-\tau}\right] \tag{62}$$

where $\gamma \equiv \frac{(1-\beta\xi_p)(1-\xi_p)}{\xi_p}$.

Linearizing the resource constraint (44), we get

$$y\hat{y}_t = c\hat{c}_t + \left[1 - \frac{1-\delta}{\mu_{z^+}\mu_\Psi}\right]\bar{k}\hat{i}_t + \frac{\bar{r}^k\bar{k}}{\mu_{z^+}\mu_\Psi}\hat{u}_t \tag{63}$$

Households Block. Linearizing (45), we get

$$\begin{aligned}\frac{(1+\phi)\lambda_w - 1}{(\lambda_w-1)(1-\beta\xi_w)}\hat{\tilde{w}}_t = -\hat{\psi}_t - \hat{\bar{w}}_t + \phi\hat{h}_t + \\ \beta\xi_w\frac{(1+\phi)\lambda_w - 1}{(\lambda_w-1)(1-\beta\xi_w)}E_{t-1}\left[\hat{\bar{w}}_{t+1} - \hat{\bar{w}}_t - \hat{\pi}_t + \hat{\pi}_{t+1} + \hat{\tilde{w}}_{t+1}\right]\end{aligned} \tag{64}$$

Linearizing (46), we get

$$\hat{\tilde{w}}_t = \frac{\xi_w}{1-\xi_w}(\hat{\pi}_t - \hat{\pi}_{t-1} + \hat{\bar{w}}_t - \hat{\bar{w}}_{t-1}). \tag{65}$$

Putting together (64) and (65), we get

$$\begin{aligned}E_{t-1}\big[\beta b_w\hat{\bar{w}}_{t+1} + \beta b_w\hat{\pi}_{t+1} - (1+(1+\beta)b_w)\,\hat{\bar{w}}_t - (1+\beta)b_w\hat{\pi}_t + \\ \phi\hat{h}_t - \hat{\psi}_t + b_w\hat{\bar{w}}_{t-1} + b_w\hat{\pi}_{t-1}\big] = 0,\end{aligned} \tag{66}$$

where $b_w \equiv \frac{\xi_w}{1-\xi_w}\frac{(1+\phi)\lambda_w-1}{(\lambda_w-1)(1-\beta\xi_w)}$.

Linearizing the law of motion of capital (47), we get

$$\hat{\bar{k}}_{t+1} = \frac{1-\delta}{\mu_{z^+}\mu_\Psi}\hat{\bar{k}}_t + \left[1 - \frac{1-\delta}{\mu_{z^+}\mu_\Psi}\right]\hat{i}_t \tag{67}$$

Linearizing (48), we get

$$\frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{i}_t + \frac{b}{y}\hat{b}_t = (1-\overline{\tau})\hat{y}_t + \frac{b}{\beta y}\left(\hat{R}_{t-1} - \hat{\pi}_t + \hat{b}_{t-1}\right) + \hat{T}_t \tag{68}$$

Linearizing (49), we get

$$\frac{\mu_{z^+}}{(\mu_{z^+}-b)(\mu_{z^+}-\beta b)}(\mu_{z^+}\hat{c}_t - b\hat{c}_{t-1}) - \frac{\beta b}{(\mu_{z^+}-b)(\mu_{z^+}-\beta b)}E_{t-1}(\mu_{z^+}\hat{c}_{t+1} - b\hat{c}_t) + \hat{\psi}_t = 0 \tag{69}$$

Linearizing (50), we get

$$\hat{\bar{r}}^k_t = \bar{r}^k \hat{u}_t \tag{70}$$

Linearizing (51), we get

$$\hat{\psi}_t = \beta E_{t-1}\left[\hat{\psi}_{t+1} - \hat{\mu}_t + \frac{\bar{r}^k \hat{\bar{r}}^k_{t+1} + \frac{1-\delta}{\mu_{z^+}\mu_\Psi}\hat{\hat{\mu}}_{t+1}}{\bar{r}^k + \frac{1-\delta}{\mu_{z^+}\mu_\Psi}}\right] \tag{71}$$

Linearizing (52), we get

$$0 = E_{t-1}\left[\hat{\hat{\mu}}_t - \kappa\mu_{z^+}\mu_\Psi(\hat{i}_t - \hat{i}_{t-1}) + \beta\kappa\mu_{z^+}\mu_\Psi(\hat{i}_{t+1} - \hat{i}_t)\right] \tag{72}$$

where $\kappa \equiv S''(\mu_{z^+}\mu_\Psi)$. Linearizing (53), we get

$$\hat{\psi}_t = E_{t-1}\hat{\psi}_{t+1} + \hat{R}_t - \hat{\pi}_{t+1} \tag{73}$$

Government Block. Finally, linearizing the Taylor rule, (54), we get

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R)\left[r_\pi \hat{\pi}_t + r_y \hat{y}_t + r_{\Delta_y}(\hat{y}_t - \hat{y}_{t-1})\right] + \bar{\varepsilon}_{R,t} \tag{74}$$

with

$$\bar{\varepsilon}_{R,t} = \rho_M \bar{\varepsilon}_{R,t-1} + u_t$$

and the government's budget constraint

$$\frac{b}{y}\hat{b}_t = \frac{b}{\beta y}(\hat{R}_{t-1} - \hat{\pi}_t + \hat{b}_{t-1}) - \tau_t - \tau\hat{y}_t + \hat{T}_t. \tag{75}$$

Timing and Information Set

As in CEE and ACEL, we assume that the period t realization of $\varepsilon_{R,t}$ is not included in the period t information set of the agents in our model. This ensures that our model satisfies the restrictions used in the VAR analysis to identify a monetary policy shock.

To take it to Dynare, we need to redefine variables in such a way that variables in period t are measurable with respect to information in period t rather than period $t-1$. To that end, we define the following variables:

$$\tilde{c}_t = E_{t-1}\hat{c}_t, \quad \tilde{w}_t = E_{t-1}\hat{\bar{w}}_t, \quad \tilde{u}_t = E_{t-1}\hat{u}_t, \quad \tilde{i}_t = E_{t-1}\hat{i}_t, \quad \tilde{\pi}_t = E_{t-1}\hat{\pi}_t, \quad \tilde{\hat{\mu}}_t = E_{t-1}\hat{\hat{\mu}}_t \tag{76}$$

The system of equations we take to Dynare is the following:

$$\hat{y}_t = \frac{y+\phi}{y}\left[(1-\alpha)\hat{h}_t + \alpha(\tilde{u}_t + \hat{\bar{k}}_t)\right] \tag{77}$$

$$\tilde{w}_t + \frac{1}{1-\alpha}\frac{y}{y+\varphi}\hat{y}_t - \hat{\hat{r}}_t^k - \frac{\alpha}{1-\alpha}\tilde{u}_t - \frac{1}{1-\alpha}\hat{k}_t = 0 \tag{78}$$

$$\hat{s}_t = \alpha\hat{\hat{r}}_t^k + (1-\alpha)\tilde{w}_t \tag{79}$$

$$\hat{\pi}_t = \frac{1}{1+\beta}\tilde{\pi}_{t-1} + \frac{\beta}{1+\beta}E_{t-1}\hat{\pi}_{t+1} + \frac{\gamma}{1+\beta}E_{t-1}\left[\hat{s}_t + \frac{\hat{\tau}_t}{1-\tau}\right] \tag{80}$$

$$y\hat{y}_t = c\tilde{c}_t + \left[1 - \frac{1-\delta}{\mu_{z^+}\mu_\Psi}\right]\bar{k}\tilde{i}_t + \frac{\bar{r}^k\bar{k}}{\mu_{z^+}\mu_\Psi}\tilde{u}_t \tag{81}$$

$$\begin{aligned}\beta\xi_w b_w \hat{\bar{w}}_{t+1} + \beta\xi_w b_w \hat{\pi}_{t+1} - \left(\lambda_w - 1 + (1+\beta)\xi_w b_w\right)\hat{\bar{w}}_t - (1+\beta)\xi_w b_w \hat{\pi}_t + \\ \phi\hat{h}_t - \hat{\psi}_t + \xi_w b_w \tilde{w}_{t-1} + \xi_w b_w \tilde{\pi}_{t-1} = 0,\end{aligned} \tag{82}$$

$$\hat{\bar{k}}_{t+1} = \frac{1-\delta}{\mu_{z^+}\mu_\Psi}\hat{\bar{k}}_t + \left[1 - \frac{1-\delta}{\mu_{z^+}\mu_\Psi}\right]\tilde{i}_t \tag{83}$$

$$\begin{aligned}&\frac{\mu_{z^+}}{(\mu_{z^+}-b)(\mu_{z^+}-\beta b)}\left(\mu_{z^+}\hat{c}_t - b\tilde{c}_{t-1}\right) \\ &\qquad - \frac{\beta b}{(\mu_{z^+}-b)(\mu_{z^+}-\beta b)}E_{t-1}(\mu_{z+}\hat{c}_{t+1} - b\hat{c}_t) + \hat{\psi}_t = 0\end{aligned} \tag{84}$$

$$\hat{\tilde{r}}_t^k = \bar{r}^k \hat{u}_t \tag{85}$$

$$\hat{\psi}_t = \beta E_{t-1} \left[\hat{\psi}_{t+1} - \hat{\tilde{\mu}}_t + \frac{\bar{r}^k \hat{\tilde{r}}_{t+1}^k + \frac{1-\delta}{\mu_{z^+}\mu_\Psi} \hat{\tilde{\mu}}_{t+1}}{\bar{r}^k + \frac{1-\delta}{\mu_{z^+}\mu_\Psi}} \right] \tag{86}$$

$$0 = \tilde{\tilde{\mu}}_t - \kappa \mu_{z^+}\mu_\Psi (\hat{\tilde{i}}_t - \tilde{\tilde{i}}_{t-1}) + \beta \kappa \mu_{z^+}\mu_\Psi (\hat{\tilde{i}}_{t+1} - \hat{\tilde{i}}_t) \tag{87}$$

$$\hat{\psi}_t = E_{t-1} \hat{\psi}_{t+1} + \hat{R}_t - \hat{\pi}_{t+1} \tag{88}$$

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R) \left[r_\pi \hat{\pi}_t + r_y \hat{y}_t + r_{\Delta_y} (\hat{y}_t - \hat{y}_{t-1}) \right] \tag{89}$$

$$b_y \hat{b}_t = \frac{R b_y}{\pi \mu_{z^+}} (\hat{R}_{t-1} - \hat{\pi}_t + \hat{b}_{t-1}) - \tau_t - \tau \hat{y}_t + \hat{T}_t \tag{90}$$

Equations (45), (47), (49) and (52) are backward-looking equations that are not used to determine the values of variables in $t = 0$, since $\hat{\pi}_0 = \hat{\tilde{w}}_0 = \hat{c}_0 = \hat{\tilde{i}}_0$. These equations are used in $t \geq 1$, once the shock is already part of the information set of agents (and there are no other shocks in the future). Hence, variables dated in t and $t+1$ do not need an information set adjustment. However, variables dated in $t-1$ do: in $t = 1$, past variables differ depending on whether the information set of $t = 0$ was used or not; for $t > 1$, since there are no more shocks, the two sets of variables coincide, so using the previous information set is inconsequential. Once this is sorted out, can can see that all other variables are static or forward-looking, so all these equations are used in $t = 0$ and onward, and some of these variables need information adjustment (like $\hat{u}_t$ in equation (42)), while others inherit that property because of equilibrium conditions (like $\hat{y}_t$ in equation (46)).

Data sources

We used the following data:

Nominal GDP: BEA Table 1.1.5 Line 1
Real GDP: BEA Table 1.1.3 Line 1
Consumption Durable: BEA Table 1.1.3 Line 4
Consumption Non Durable: BEA Table 1.1.3 Line 5
Consumption Services: BEA Table 1.1.3 Line 6
Private Investment: BEA Table 1.1.3 Line 7
GDP Deflator: BEA Table 1.1.9 Line 1
Capacity Utilization: FRED CUMFNS
Hours Worked: FRED HOANBS
Nominal Hourly Compensation: FRED COMPNFB
Civilian Labor Force: FRED CNP16OV
Nominal Revenues: BEA Table 3.1 Line 1

Nominal Expenditures: BEA Table 3.1 Line 21
Nominal Transfers: BEA Table 3.1 Line 22
Nominal Gov't Investment: BEA Table 3.1 Line 39
Nominal Consumption of Net Capital: BEA Table 3.1 Line 42
Effective Federal Funds Rate (FF): FRED FEDFUNDS
Market Value of Government Debt: Hall et al. (2018)

Parameters value

We divide the parameter set into two groups. For the parameters in the first group, we calibrate them following Christiano et al. (2010). For those in the second group, we estimate them using IRF matching methods.

The next table shows the values of the parameters we calibrate.

Parameter	Value	Description
α	0.36	Capital share
ξ_w	0.66	Wage stickiness
δ	0.025	Quarterly depreciation rate
β	0.9926	Quarterly discount factor
λ_w	1.05	Wage markup
μ_z	1.0041	Quarterly gross neutral technology growth
μ_Ψ	1.0018	Quarterly gross investment technology growth
π	1.0083	Steady-state quarterly gross inflation rate
τ	0.27	Steady-state sales tax
g_y	0.2	Steady-state government consumption to GDP ratio
b_y	0.75	Steady-state government debt to GDP ratio

The next table shows the values of the estimated parameters.

Parameter	Value	Description
ξ_p	0.8364	Price stickiness
λ_f	1.01	Price markup
σ_a	0.2776	Capacity adjustment costs curvature
κ	13.6947	Investment adjustment cost curvature
b	0.8067	Consumption habit
ϕ	0.2919	Inverse of Frisch elasticity
ρ_R	0.8342	Taylor rule: Interest smoothing
r_π	1.2315	Taylor rule: Inflation coefficient
r_y	0.1230	Taylor rule: GDP coefficient
$r_{\Delta y}$	0.9503	Taylor rule: GDP growth coefficient
ρ_M	0.0685	Autocorrelation monetary shock

D.1 Model reestimation

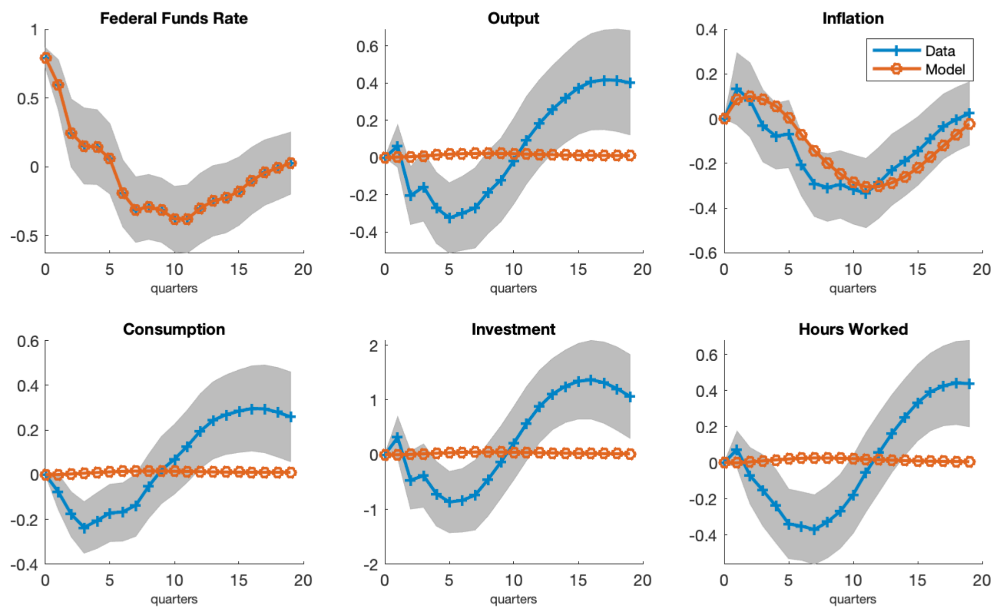

FIGURE 8: Model impulse response functions to a monetary shock. Parameters re-estimated using fiscal data. Gray area represents 95% confidence intervals.

E Quantitative TANK model

The system of equations characterizing the log-linear approximation of the equilibrium of the economy is isomorphic to the one in RANK, except for three differences:

1. aggregate consumption is the sum of the consumption of borrowers and savers

$$\hat{c}_t = \chi \hat{c}_t^s + (1-\chi)\hat{c}_t^b$$

   where we have imposed that steady-state transfers are such that $c^b = c^s = c$;

2. the unions' objective is to maximize a weighted average of the utility of borrowers and savers. Borrowers and savers need to supply the same amount of labor, and the marginal value of the wage used by unions in their optimization problem is

$$\hat{\psi}_t = \chi \hat{\psi}_t^s + (1-\chi)\hat{\psi}_t^b.$$

3. both types of households face the same Euler equation, given by equation (69), but while the optimality condition of bonds for savers is given by (73), the borrowers' condition is given by

$$\hat{\psi}_t^b = E_{t-1}\left[\hat{\psi}_{t+1}^b + \hat{R}_t^b - \hat{\pi}_{t+1} + \eta \hat{d}_t\right]$$

   where $\eta \equiv \phi_{11}(1,1) + \phi_{12}(1,1)$, the interest rate of the borrowers satisfies

$$\hat{R}_t^b = \hat{R}_t + \varphi \hat{d}_t$$

   where $\varphi \equiv \phi_2(1,1)$, and their budget constraint is

$$\frac{c}{y}\hat{c}_t^b = \frac{\overline{w}h}{y}(\hat{\overline{w}}_t + \hat{h}_t) + \frac{\overline{d}_y}{\beta}\left(\hat{R}_{t-1}^b - \hat{\pi}_t + d_{t-1}\right) - \overline{d}_y d_t$$

   where $\overline{d}_y \equiv \frac{\overline{d}}{y}$ and we have assumed that $\hat{T}_t^b = 0$.

To keep the timing consistent with our VAR identification, we make the same timing assumptions as in the RANK version of the model.

The next table shows the values of the parameters we calibrate.

Parameter	Value	Description
α	0.36	Capital share
ξ_w	0.66	Wage stickiness
δ	0.025	Quarterly depreciation rate
β	0.9926	Quarterly discount factor
λ_w	1.05	Wage markup
μ_z	1.0041	Quarterly gross neutral technology growth
μ_Ψ	1.0018	Quarterly gross investment technology growth
π	1.0083	Steady-state quarterly gross inflation rate
τ	0.27	Steady-state sales tax
g_y	0.2	Steady-state government consumption to GDP ratio
b_y	0.75	Steady-state government debt to annual GDP ratio
χ	5/6	Fraction of savers
$\overline{d}_y$	0.40	Steady-state household debt to annual GDP ratio

The next table shows the values of the estimated parameters.

Parameter	Value	Description
ξ_p	0.8629	Price stickiness
λ_f	1.5	Price markup
σ_a	4.2837	Capacity adjustment costs curvature
κ	8.1487	Investment adjustment cost curvature
b	0.7774	Consumption habit
ϕ	0.1590	Inverse of Frisch elasticity
η	0	Borrowing rate curvature
φ	0.1509	Borrowing rate slope

Our estimation of λ_f hits our imposed upper bound of 1.5. When we relaxed this constraint, λ_f would increase to implausible values, with a minor impact on the likelihood value and the value of other parameters. Given that the likelihood function is relatively flat in λ_f, we decided to report the value for 1.5.