Sowing the Seeds of Financial Crises: Endogenous Asset Creation and Adverse Selection

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Abstract

What sows the seeds of financial crises, and what policies can help avoid them? I model the interaction between the *ex-ante* production of assets and *ex-post* adverse selection in financial markets. Positive shocks that increase market prices exacerbate the production of low-quality assets and can increase the likelihood of a financial market collapse. The interest rate and the liquidity premium are endogenous and depend on the functioning of financial markets as well as the total supply of assets (private and public). Optimal policy balances the economy's liquidity needs *ex-post* with the production incentives *ex-ante*, and it can be implemented with three instruments: government bonds, asset purchase programs, and transaction taxes. Public liquidity improves incentives but implies a higher deadweight loss than private market interventions. Optimal policy does not rule out private market collapses but mitigates the fluctuations in total liquidity.

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1 Introduction

It is widely believed that the recession that hit the US economy in 2008 originated in the financial sector. The years previous to the Great Recession were characterized by a rapid increase in the private production of assets that were considered safe, mostly through securitization. Many of the markets for these assets later collapsed, marking the starting point of the deepest recession in the post-war era. To mitigate the adverse consequences of the financial crisis, the Fed responded aggressively using a variety of instruments, such as the direct purchase of mortgage-back securities (MBS) as part of the Quantitative Easing (QE) 1 program in 2008, as well as the active management of public liquidity, with the purchase of long-term Treasuries as part of QE 2 in 2010.[1] However, relatively little theoretical work has analyzed the optimal policy *mix* in economies exposed to financial distress. This is particularly important in contexts in which policy can be a contributing factor to sowing the seeds of the next crisis. For example, policies aimed at improving the efficiency of private markets might induce excessive risk build-ups, while policies targeted at increasing public liquidity can be costly and even exacerbate the malfunctioning of private markets by crowding out the private sector. Therefore, a formal analysis requires understanding the interplay between the frictions in the economy and the dynamic effects of interventions.

In this paper, I build a model of an economy susceptible to risk build-ups that can lead to financial crises and use it as a laboratory to study the role of policy in improving market outcomes. The model features an endogenous determination of *financial market fragility*, i.e., the probability of a discontinuous drop in the volume traded in private financial markets, which disrupts the flow of funds to the agents with the highest valuation. I analyze constrained efficiency by considering the problem of a planner who faces the same constraints as the private economy. I show that a government can implement the constrained efficient allocation by a combination of three instruments: government bonds, asset purchase programs, and transaction (or "Tobin") taxes.

I develop a model of asset quality determination in which the ex-ante production of assets interacts with ex-post adverse selection in financial markets. Agents in the economy face idiosyncratic risk, and financial markets are incomplete. Assets play a dual role and derive their value from the dividends they pay and the liquidity services they provide. Better-quality assets pay higher dividends, but because of adverse selection in markets, they sell at a pooling price with lower-quality assets. This cross-subsidization between high- and low-quality assets introduces a motive for agents to produce more lemons when they expect prices to be high, since they expect to sell the assets rather than keep them until maturity. As a consequence, the theory predicts that the production of low-quality assets is more responsive to market conditions than that of high-quality assets. Shocks that improve the functioning of financial markets or increase their scarcity

[1] Since the Great Recession, the Fed has adopted these instruments as part of its standard toolkit to deal with large shocks. For example, in 2019, the New York Fed increased its repo operations in response to a sudden spike of the repo rate in September of that year. In March of 2020, the Fed implemented a new round of securities purchases to mitigate the financial turmoil caused by the COVID-19 crisis.

value exacerbate the production of lemons and may even increase the exposure of the economy to a financial market collapse –a process that disrupts liquidity.

An important distinctive feature of the model is that the supplies of privately produced tradable assets and government bonds (i.e., private and public liquidity) interact through an endogenously determined *liquidity premium*. Because of the market incompleteness, the allocation of resources in the competitive equilibrium is always imperfect, and the liquidity premium is a measure of the *scarcity* of assets that can facilitate the flow of funds. Private and public assets can fulfill this role, but only private assets suffer from adverse selection. My theory predicts that reductions in the supply of government bonds increase the production of private assets that can serve as substitutes. But because low-quality assets are more sensitive to changes in the value of liquidity services, the private production is biased towards low-quality assets. Hence, a shortage of safe assets induces a deterioration of private asset quality. Indeed, my model predicts that the reductions in US government bonds in the late 1990s due to sustained fiscal surpluses, as well as the increased foreign demand for US-produced safe assets in the early 2000s (a consequence of the so-called "savings glut"), generated perverse effects on the quality composition of privately produced assets.[2]

The mechanics of the model hinge upon the agents' valuation of the different asset qualities. Agents with high-quality assets sell them only if their liquidity needs are high relative to the price discount they suffer in the market due to the adverse selection problem. In contrast, agents with low-quality assets always sell their holdings. Anticipating that this will be their strategy in the market, agents adjust their quality production decisions to the expected market conditions. If the market's expectations are high –in the sense that expected prices are high– agents anticipate that the probability they will sell their assets is relatively high, independent of their quality. In this case, more low-quality assets are produced. That is, low-quality assets are produced for *speculative motives*: not for their fundamental value, but for the profit the agent can make from selling in the market. This result is an extension of Akerlof (1970), which shows that the decision to sell non-lemons is more sensitive to prices than the decision to sell lemons. In my model, Akerlof's result still holds in the market for assets. But the lower exposure of the private valuation of high-quality assets to market conditions results in the opposite sensitivity in the production stage.

While the theory presented is silent about the specifics of the safe asset production process, the economic forces it highlights are typical of the entire process of transforming illiquid assets into liquid ones. Safety refers to a characteristic of assets that are perceived as high quality, have an active (*liquid*) market, and facilitate financial transactions.[3] While traditionally this characteristic was mostly limited to government bonds and bank deposits, in the last 30 years there has been a large increase in the use of other privately produced assets, such as asset- and mortgage-backed securities (see Gorton et al. (2012)). This process was particularly stark in the mortgage market,

[2]See Caballero (2006) and Caballero (2010) for a discussion of safe asset shortages. For a quantitative analysis, see Barclays Capital (2012) and Caballero et al. (2017).

[3]This has been recently emphasized, for instance, by Calvo (2013), Gorton et al. (2012) and Gorton (2017).

which saw an explosion of non-standard, low-documentation mortgages, and low credit score borrowers.[4] In fact, the Bank for International Settlements (2001) articulated an early warning about the deterioration of the quality of assets used as collateral. In my interpretation, the production of assets comprises both the origination of loans (e.g., mortgages) and their posterior securitization (e.g., AAA-rated private-label mortgage-backed securities).[5] In both cases, the "producers" know more than other market participants about the underlying quality of these products, either because they have collected information that cannot be credibly transmitted or because they know how much effort they put into the process. Hence, the problem of quality production and adverse selection can be present in the whole intermediation chain.[6]

I then solve the problem of a planner who faces the same constraints as the private economy. Solving for the constrained optimal allocation in the presence of moral hazard, adverse selection, and aggregate risk is a complex task. A key step in the solution method is an equivalence result of the planner's problem in terms of allocations subject to resource and information constraints, and a Ramsey problem where the planner has only access to state-contingent government bonds, transaction subsidies, and taxes. This equivalence significantly reduces the dimensionality of the problem. Still, the Ramsey program presents several challenges that need to be overcome. First, the presence of aggregate risk implies that the optimal policy involves functions of the aggregate state. Second, the discontinuities in the private market generate kinks in the planner's problem, which implies that the solution is not fully characterized by first-order conditions. Third, because the planner internalizes the effects that its actions have on the agents' incentives, its calculations involve a two-way interaction in the induced equilibrium of the economy across different dates and states.

I find that the optimal policy balances two forces: a *liquidity effect* and an *incentives effect*. The liquidity effect captures the impact of the planner's plan on the reallocation of resources for a given composition of asset quality. This is the force that justifies government intervention in models in which the private sector fails to fully reallocate resources to those agents with the highest valuations, as in Woodford (1990) and Holmström and Tirole (1998). The liquidity effect is static: since it takes the asset quality composition as given, it is optimized state by state and does not incorporate intertemporal feedback effects. In contrast, the incentives effect captures how (expected) changes in market conditions *ex-post* affect the incentives to produce asset quality *ex-ante*. This effect is specific to the problem with endogenous asset quality production and moral hazard. Moreover,

[4] See Ashcraft and Schuermann (2008). While origination of non-agency mortgages (subprime, Alt-A, and Jumbo) was \$680 billion in 2001, it increased to \$1,480 billion in 2006, a growth of 118%. In contrast, origination of agency (prime) mortgages decreased by 27%, from \$1443 billion in 2001 to \$1040 billion in 2006. Moreover, while only 35% of non-agency mortgages were securitized in 2001, that figure grew to 77% in 2006.

[5] An important question is whether tranching can help avoid adverse selection. If the balance sheets of financial intermediaries are difficult to monitor, then intermediaries can always go back to the market to sell any remaining fraction of assets, limiting the role for "skin-in-the-game." Moreover, certification by third parties (e.g., rating agencies) can have limited success if agents learn to game the rating models, or if the incentives of the third party are compromised.

[6] There is an empirical literature that measures the extent of adverse selection in financial markets (see, e.g., Keys et al. (2010), Demiroglu and James (2012), Downing et al. (2009), Krainer and Laderman (2014), and Piskorski et al. (2015)).

the incentives effect is dynamic: changes in market conditions and the total liquidity available affect the private decisions to produce asset quality, affecting the equilibrium in the economy in *all* possible states in the future, which feeds back into the optimal policy decisions.

Notably, the different policy instruments interact. Public provision of liquidity and private market interventions affect the economy through similar channels. Both can increase the total available liquidity and shape the incentives to produce asset quality. However, the two instruments differ in important ways. Interventions aimed at restoring private market functioning require an increase in the price received by sellers, which tends to increase the incentives to produce low-quality assets. In contrast, direct liquidity provisions reduce the liquidity premium and therefore reduce the incentives to produce low-quality assets, but are more costly than private market interventions. I find that the optimal policy prescribes an aggressive increase in the supply of public liquidity in times of crisis, i.e., when private markets collapse, while it aims to stabilize market prices, interest rates, and trading volume in normal times (in a *leaning against the wind* type of policy) and provide only a small amount of public liquidity. In fact, reducing the probability of a (private) financial market collapse is not an objective of the planner *per se*, as (private) market fragility can be higher under the optimal policy than in the *laissez-faire* equilibrium. Instead, the planner tries to mitigate variation in the *total liquidity* in the economy (private and public) across states, trading off between the cost of interventions and the incentives the policies generate.

Finally, I present a novel equivalence result between transaction subsidies and asset purchase programs. This is important from a policy point of view, as transaction subsidies can generate spurious trades aimed exclusively at collecting the subsidy, defeating the purpose of the instrument. Asset purchase programs do not suffer from this problem. Moreover, asset purchase programs are part of the toolkit recently used by the Fed to improve the liquidity in private markets after the Great Recession, so their study can be of independent interest.

Literature Review. This paper contributes to several strands of the literature. First, the paper is related to the literature that incorporates adverse selection in financial markets into macroeconomic models. An early paper that studies this problem is Eisfeldt (2004). Kurlat (2013) and Bigio (2015) build a model in which adverse selection in financial markets is used to explain the sudden collapse of the market for mortgage-related securities during the Great Recession. However, these papers take the distribution of asset quality as exogenously given and abstract from the role of government bonds as a source of liquidity. My paper builds on these insights but, taking a step back, focuses on how the endogenous determination of asset quality distribution interacts with the state of the economy and the policy stance. This extension is key to understanding the build-ups of risks emphasized in these papers, as well as for the design of the optimal policy plan.

There is also a related literature that explores the interaction between the incentives to produce asset quality and ex-post adverse selection in financial markets (see, for example, Parlour and Plantin (2008), Chemla and Hennessy (2014), Vanasco (2017)). I contribute to this literature by studying the interaction between the private and public liquidity provision in a setting with aggregate risk. In my model, the incentives to produce private assets interact with the supply of public assets through an endogenously determined liquidity premium, and optimal policy leverages this relationship. Moreover, the presence of aggregate risk introduces an endogenously determined probability of a financial crisis and allows me to explore state-contingent policies that distinguish between "normal" and "crisis" states. In particular, I identify the conditions under which it is optimal to aggressively increase the supply of public liquidity (as with QE 2), when it is optimal to support the private markets (as with QE 1), or whether the private markets should be taxed. Contemporaneous work by Fukui (2018) and Neuhann (2017) also study how moral hazard and adverse selection in private markets can lead to boom-bust dynamics. Fukui (2018) focuses on the reallocation of physical capital while Neuhann (2017) studies changes in the demand for financial assets triggered by variations in the distribution of wealth. In contrast, I study the endogenous determination of the liquidity premium and the interaction between the private and public liquidity provision. This distinction is crucial for the policy analysis.

My focus on the public provision of liquidity is shared by a large body of literature that emphasizes the role of government bonds in facilitating the flow of resources among agents in economies with financial frictions. Woodford (1990) shows that when agents face binding borrowing constraints, a higher supply of government bonds can increase welfare. Holmström and Tirole (1998) also highlight the role of tradable instruments when agents cannot fully pledge their future income. Geromichalos et al. (2007) is one of the first papers to study the effect of monetary policy on asset prices in a monetary-search environment. They show that money can increase welfare when the supply of private "tradable" assets is insufficient to satiate the agents' liquidity needs. Gorton and Ordoñez (2013) also study the interaction between public and private liquidity, but their focus is on the production of information, whereas my model highlights the liquidity premium and the production of asset quality.[7]

Finally, this paper is related to the literature that studies the scarcity of safe assets more generally. Kiyotaki and Moore (2012) and Del Negro et al. (2017) consider the adverse effects of an exogenous reduction in the collateral value of private assets.[8] Del Negro et al. (2017) argue that the Fed's aggressive response in 2008, which substantially increased the supply of public liquidity, helped avoid a deeper recession. My paper complements their analysis by microfounding the source of the private market deterioration and by studying the optimal policy mix when both public provision of liquidity and private market interventions are available. Consistent with their findings, I find that, in the event of a crisis, an aggressive policy of providing public liquidity is optimal. However, there are important differences. First, the optimal policy mix also includes interventions in the private markets. Second, while Del Negro et al. (2017) focus on *ex-post* poli-

[7] A significant number of papers have documented that private production of safe assets increases when the supply of government bonds is low (and vice versa). See, e.g., Gorton et al. (2012), Krishnamurthy and Vissing-Jorgensen (2015), Greenwood et al. (2015) and Sunderam (2015). Krishnamurthy and Vissing-Jorgensen (2012) show that an increase in the supply of government bonds reduces the liquidity premium.

[8] Caballero and Farhi (2018) study the effects of an exogenous reduction in the supply of safe assets in an economy with sticky prices.

cies, my analysis includes the *ex-ante* incentives that contribute to risk build-ups. In this sense, the trade-offs in the economy are fundamentally different. Relatedly, Tirole (2012) and Philippon and Skreta (2012) study the optimal policy in a setting in which private markets have collapsed due to an adverse selection problem. However, they focus on ex-post interventions in the private markets, while I study the ex-ante problem. Additionally, I consider the public provision of liquidity as an additional instrument. Angeletos et al. (2016) study the role of liquidity but abstract from asymmetric information and the possibility of financial crises. Also close to the exercise in this paper is Jeanne and Korinek (2020), who study the role of *ex-ante* (macroprudential) policies and *ex-post* (liquidity) policies jointly in a model with pecuniary externalities. Instead, I focus on economies where asymmetric information is the key friction.

Outline. The rest of the paper is organized as follows. Section 2 presents the model. Section 3 studies the equilibrium determination and its positive implications, including the economy's response to changes in the supply of public liquidity. The normative analysis is developed in Section 4. Section 5 concludes. All the proofs are presented in the appendix.

2 The Model

The economy lasts for three periods and is populated by a measure one of ex-ante identical agents. Agents choose the quality of the assets they produce, anticipating that in the future they will face a "liquidity shock" that affects their intertemporal preference for consumption and a market for private assets that suffers from adverse selection. Agents can also trade government bonds.

2.1 The Environment

Agents. There are three dates, 0, 1, and 2, and two types of goods: a final consumption good and Lucas trees. The economy is populated by a measure one of agents. Agents receive an endowment of final consumption good of $W_t > 0$ in period $t = 0, 1, 2$. In period 0, they operate a technology that transforms final consumption goods into trees, which pay a dividend in period 2.[9]

Agents' preferences are given by

$$U = E\left[\mu c_1 + c_2\right],$$

where c_t denotes consumption in $t = 1, 2$, μ is a random idiosyncratic "liquidity shock" (uncorrelated across agents), which is their private information, and the expectation is taken with respect to μ and an aggregate state, described below. The liquidity shock affects the agents' marginal utility of consumption in period 1. From period 0 point of view, μ is distributed according to the cumulative distribution function $G(\mu)$ in $[1, \mu^{\max}]$ with associated continuous density $g(\cdot)$.

[9] A Lucas tree in this economy is a technology that delivers an exogenous dividend in period 2. The trees stand for a privately produced asset, in contrast to publicly supplied assets, i.e., government bonds.

Technology. Agents have access to a technology to produce trees in period 0. There are two types of trees. An agent can transform h_B units of the consumption good into h_B units of low-quality, or "bad," trees, and $C(h_G)$ units of the consumption good into h_G units of high-quality, or "good," trees, where $C(0) = 0$, $C'(\cdot) \geq 1$ and $C''(\cdot) > 0$. Let λ_E denote the fraction of good trees in the economy in period 1, that is $\lambda_E \equiv \frac{H_G}{H_G+H_B}$, where H_G and H_B denote the aggregate stock of good and bad trees, respectively.

Trees deliver fruit in the form of final consumption good in period 2. A unit of good tree pays Z with certainty at maturity. In contrast, only a fraction α of bad trees deliver fruit in period 2, so that the expected payoff of a unit of bad tree is αZ.[10] The fraction of bad trees that deliver fruit is known one period in advance. Thus, in period 1, the fraction α is common knowledge. However, in period 0 agents believe that α is a random variable distributed according to the cumulative distribution function F in the interval $[\underline{\alpha}, \overline{\alpha}] \subseteq [0, 1]$. One can interpret α as an aggregate shock to the productivity of bad trees, so that a higher α implies a higher quality of bad trees, or $1 - \alpha$ as a default rate of bad trees in period 2. I assume that F is continuous and non-degenerate, with associated continuous density $f(\cdot)$. Moreover, only the owner of the tree can determine its quality. This will be important when I describe the financial markets below.

Financial Markets. Due to the idiosyncratic liquidity risk in period 1, there are gains from trade in this economy. I assume that financial markets are incomplete. In particular, I assume that agents can trade in only two markets: i) a market for the trees produced in period 0, and ii) a market for government bonds. These markets can be interpreted as a metaphor for collateralized debt markets, like "repos" or short-term commercial paper.[11]

I follow Kurlat (2013) and Bigio (2015) and assume that there is a unique market in which all tree qualities are traded, that buyers cannot distinguish the quality of a specific unit of tree but can predict what fraction of each type there is in the market, and that the market is anonymous, non-exclusive and competitive. These assumptions imply that the market features a pooling price, P_M. Buyers get a diversified pool of trees from the market, where λ_M is the fraction of good trees in the pool.

To make the distinction between good and bad trees stark, I make the following assumption.

Assumption 1. *The expected payoff of the trees satisfies:*

$$\frac{Z}{C'(C^{-1}(W_0))} > E[\alpha Z].$$

Assumption 1 implies that if the quality of trees were observable, the return of bad trees would be lower than the return of good trees for all relevant production scales. Thus, in an economy with

[10] Alternatively, one can assume that each bad trees pays αZ. Both assumptions are equivalent in this model.

[11] Bigio (2015) presents an equivalence result between a market for trading assets and a repo contract when there is no cost of defaulting besides delivering the collateral to the creditor. This is a standard assumption in papers on collateralized debt. See, for example, Geanakoplos (2010) and Simsek (2013).

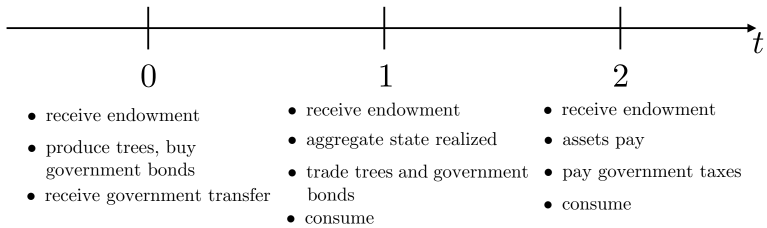

Figure 1: Timing

perfect information, bad trees would not be produced.

Government. In period 0, the government supplies bonds, which mature in period 2. The government's budget constraint in period 0 is

$$T_0 = Q_0^B B_0,$$

where Q_0^B denotes the price of government bonds, B_0 is the bond supply, and T_0 is a lump-sum transfer. The budget constraint in period 2 is

$$T_2 + B_0 = 0,$$

where T_2 is a lump-sum transfer in period 2.

Aggregate State and Timing. The exogenous state of the economy is given by the distribution of liquidity shocks in the population and the realized quality of bad trees, α. The endogenous state is given by the cross-section distribution of assets and shocks across agents. As a consequence of the linearity of the agents' preferences and constraints (see programs (P0), (P1) and (P2) below), prices and aggregate quantities do not depend on the distribution of portfolios in the population. Therefore, the relevant state in periods 1 and 2 is $X \equiv \{\alpha, \lambda_E, B_0\}$, where the total number of trees in the economy, $H \equiv H_G + H_B$, can be obtained as the unique solution to $H = W_0 - C(\lambda_E H) + \lambda_E H$ for a given λ_E.

To summarize, the timing of the economy is as follows. Agents start period 0 with an endowment of the final consumption good W_0, they receive a lump-sum transfer T_0, and they decide how to allocate their wealth between the production of private assets (good and bad) and government bonds. In period 1, agents receive an endowment of the final consumption good W_1, the aggregate shock α is realized, and agents receive an idiosyncratic liquidity shock, μ. Agents choose between two possible uses of the consumption goods they hold, that is, their *liquid wealth*: to consume or to buy assets in the market (trees and government bonds). Finally, in period 2, agents receive an endowment W_2, all assets pay, and agents consume. Figure 1 summarizes the timing.

2.2 First Best

Let's consider first the (*ex-ante*) Pareto efficient allocation. Suppose that the planner can observe the individual agents' realization of μ, and it chooses each agent's consumption, $c_1(\mu, \alpha)$ and $c_2(\mu, \alpha)$, as a function of the agent's idiosyncratic shock μ and the aggregate state α. Moreover, assume that the planner can determine the production of tree quality in period 0. Therefore, the planner's problem is given by

$$\max_{\{c_1(\mu,\alpha), c_2(\mu,\alpha)\}, H_G, H_B} E\left[\mu c_1(\mu, \alpha) + c_2(\mu, \alpha)\right] \tag{FB}$$

subject to

$$C(H_G) + H_B = W_0$$

$$\int_1^{\mu^{\max}} c_1(\mu, \alpha) dG(\mu) = W_1$$

$$\int_1^{\mu^{\max}} c_2(\mu, \alpha) dG(\mu) = W_2 + ZH_G + \alpha Z H_B$$

The next proposition characterizes the solution.

Proposition 1 (First Best)**.** *In the Pareto optimal allocation, only good trees are produced and only the agents with the highest realization of* μ, $\mu = \mu^{\max}$, *consume in period* 1. *Any allocation of consumption in period* 2 *is consistent with Pareto optimality.*

There are two dimensions to the planner's problem. On the one hand, the planner seeks to achieve *production efficiency*; that is, it makes sure that only good trees are produced in period 0. On the other hand, it also aims for *consumption efficiency*, by allocating the endowment W_1 to the agents that value it the most in period 1.

Program (FB) is very demanding in terms of the information available to the planner. It assumes that the planner can observe μ and make transfers conditional on this information, while also choosing the quality of trees produced by the agents. In Appendix B, I revisit the planner's problem under different assumptions about the information restrictions to better understand the role of the frictions in this economy.

2.3 Agents' Problem

The agents' problem in period 2 is simple: they receive the endowment W_2, collect the dividends from the trees and government bonds they own, pay taxes and consume. Their value function is

$$V_2(h_G, h_B, b; X) = W_2 + Zh_G + \alpha Z h_B + b + T_2(X), \tag{P2}$$

where b denotes the holdings of government bonds.

Let's turn to period 1. Denote the purchases of trees in the market by m. If an agent buys m

units of trees, a fraction λ_M of them is good, while a fraction $1 - \lambda_M$ is bad.[12] Let s_G and s_B denote the sales of good and bad trees, respectively. The agents' problem in state X is given by:

$$V_1(h_G, h_B, b; \mu, X) = \max_{\substack{c,m,s_G,s_B,\\ h'_G,h'_B,b'}} \mu c + V_2(h'_G, h'_B, b'; X), \tag{P1}$$

subject to

$$c + P_M(X)(m - s_G - s_B) + Q_1^B(X)(b' - b) \leq W_1, \tag{1}$$

$$h'_G = h_G + \lambda_M(X)m - s_G, \tag{2}$$

$$h'_B = h_B + (1 - \lambda_M(X))m - s_B, \tag{3}$$

$$c \geq 0, \quad m \geq 0, \quad b' \geq 0, \quad s_G \in [0, h_G], \quad s_B \in [0, h_B],$$

where P_M is the price of one unit of a tree and Q_1^B is the price of government bonds in period 1. Constraint (1) is the agent's budget constraint, which states that consumption plus net purchases of assets (trees and government bonds) cannot be larger than the endowment W_1. Constraints (2) and (3) are the laws of motion of good and bad trees, respectively, which are given by the agents' initial holdings of trees plus a fraction of the purchases they make (where the fraction is given by the market composition of each type of tree) minus the sales they make.

The linear structure of the problem implies that we can characterize the agents' decisions in period 1 by two thresholds on μ: μ_B, which determines whether to consume or buy assets, and μ_S, which determines whether to sell good trees. The intuition is simple. The return from buying assets in the market is given by $\frac{\lambda_M Z + (1-\lambda_M)\alpha Z}{P_M}$ for trees and $\frac{1}{Q_1^B}$ for government bonds, which is the same for all agents. In equilibrium, market clearing requires that $\frac{\lambda_M Z + (1-\lambda_M)\alpha Z}{P_M} = \frac{1}{Q_1^B} \equiv r_M$. Because the utility from consuming in period 1 and the return from the market are both linear, agents simply compare μ and r_M to decide whether to use their liquid wealth to consume or to buy assets. Thus, the threshold for consumption satisfies $\mu_B = r_M$.

The decision to sell good trees involves similar calculations. In equilibrium, the market price of trees is always below the fundamental value of good trees, $\frac{Z}{r_M}$. Hence, the only reason the agent would sell her good trees is if the utility derived from consuming in period 1 instead of period 2 compensates for the loss. This happens if $\mu > \mu_S$, where $\mu_S \equiv \frac{Z}{P_M} \geq \mu_B$. Note that, in this economy, all agents sell their bad trees. Figure 2 summarizes these choices.

An important result that will significantly simplify the analysis that follows is the linearity of the agents' value function with respect to their holdings of each type of tree and government bonds.

[12] More formally, λ_M should denote the agents' beliefs about the quality of the trees in the market. Since I focus on *Rational Expectations Equilibria*, λ_M will coincide with the actual quality in the market. To save on notation, I have already imposed this equilibrium condition.

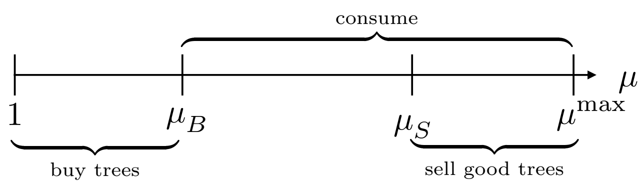

Figure 2: Agents' choices in period 1

Lemma 1. *The agents' value function in period* 1*,* $V_1(h_G, h_B, b; \mu, X)$*, is linear:*

$$V_1(h_G, h_b, b; \mu, X) = \tilde{\gamma}(\mu, X)W_1 + \tilde{\gamma}_G(\mu, X)h_G + \tilde{\gamma}_B(\mu, X)h_B + \tilde{\gamma}_{GB}(\mu, X)b + W_2 + T_2(X),$$

where

$$\tilde{\gamma}(\mu, X) = \max\{\mu, \mu_B(X)\}, \quad \tilde{\gamma}_{GB}(\mu, X) = \frac{\tilde{\gamma}(\mu, X)}{\mu_B(X)},$$
$$\tilde{\gamma}_G(\mu, X) = \max\{\mu P_M(X), Z\}, \quad \tilde{\gamma}_B(\mu, X) = \tilde{\gamma}(\mu, X)P_M(X). \tag{4}$$

Lemma 1 follows directly from the linearity of the objective function and the constraints. The agents' marginal utility of an extra unit of consumption good in period 1 is given by $\tilde{\gamma}(\mu, X)$, which compares the marginal utility of consumption with the return from the market. Then, the value of a unit of government bond is $\tilde{\gamma}(\mu, X)Q_1^B(X) = \frac{\tilde{\gamma}(\mu,X)}{\mu_B(X)}$. Let's turn to the value of trees. Since bad trees are always sold, holding one unit of bad tree delivers $\tilde{\gamma}(\mu, X)P_M(X)$. In contrast, good trees are sold only if the utility from selling in period 1, $\mu P_M(X)$, is higher than the utility from keeping it until maturity, Z. Note that the value of bad trees does not directly depend on its payoff in period 2, since no agent who *starts* the period owning bad trees holds them until maturity.

Finally, the problem of an agent in period 0 is given by

$$V_0 = \max_{h_G, h_B, b} E[V_1(h_G, h_B, b; \mu, X)], \tag{P0}$$

subject to

$$C(h_G) + h_B + Q_0^B b \leq W_0 + T_0, \tag{5}$$
$$h_G \geq 0, \quad h_B \geq 0, \quad b \geq 0,$$

where Q_0^B is the price of government bonds in period 0, T_0 is a lump-sum transfer, and E denotes the expectation operator with respect to μ and α. Constraint (5) is the agents' budget constraint, which states that expenditures in the production of trees and purchases of government bonds cannot be larger than the endowment plus transfers, $W_0 + T_0$.

Before solving the agents' problem in period 0, it is useful to define the key objects in the analysis that follows: the shadow value of trees.

Definition 1 (Shadow Value of Trees)**.** *The shadow value of good and bad trees are given by*

$$\gamma_G \equiv E\left[\widetilde{\gamma}_G(\mu, X)\right] = E\left[\max\left\{\mu P_M(X), Z\right\}\right],$$
$$\gamma_B \equiv E\left[\widetilde{\gamma}_B(\mu, X)\right] = E\left[\max\{\mu, \mu_B(X)\} P_M(X)\right].$$

The shadow value of trees is the expected value of the marginal utility of the trees in period 1, given by (4). To understand the intuition behind these expressions, I decompose them into three elements: a fundamental value, a liquidity premium, and an adverse selection tax/subsidy:

$$\gamma_G = E\left[\underbrace{Z}_{\text{fund. value}} + \underbrace{\left(\frac{\widetilde{\gamma}(\mu, X)}{r_M(X)} - 1\right) Z}_{\text{liq. premium}} - \min\left\{\widetilde{\gamma}(\mu, X)\underbrace{\left(\frac{Z}{r_M(X)} - P_M(X)\right)}_{\text{adv. sel. tax}}, \left(\frac{\widetilde{\gamma}(\mu, X)}{r_M(X)} - 1\right) Z\right\}\right], \tag{6}$$

$$\gamma_B = E\left[\underbrace{\alpha Z}_{\text{fund. value}} + \underbrace{\left(\frac{\widetilde{\gamma}(\mu, X)}{r_M(X)} - 1\right) \alpha Z}_{\text{liq. premium}} + \widetilde{\gamma}(\mu, X)\underbrace{\left(P_M(X) - \frac{\alpha Z}{r_M(X)}\right)}_{\text{adv. sel. subs.}}\right]. \tag{7}$$

First, the fundamental value is given by the dividend each type of tree pays in period 2, which is Z for good trees and αZ for bad trees.[13] Second, trees in this economy derive value from the fact that they can be traded in period 1, transforming a dividend in period 2 into resources in period 1, when they are potentially more valuable to the owner. The liquidity premium is a consequence of the *liquidity services* tradeable assets provide in economies with incomplete markets, as emphasized by Holmström and Tirole (2001). Note that in the first best, $\mu_B(X) = \mu^{\max}$ and, therefore, the liquidity premium would equal zero. A crucial feature of the analysis that follows, particularly the normative implications of Section 4, rely on the endogeneity of the liquidity premium.

Finally, the asymmetric information problem in the market for trees introduces a wedge that is negative for good trees and positive for bad trees. Since the market price of trees is always between the fundamental value of good and bad trees in period 1, that is, $P_M(X) \in \left[\frac{\alpha Z}{r_M(X)}, \frac{Z}{r_M(X)}\right]$, the good trees feature an *adverse selection tax*. However, this tax is charged only if the tree is sold. Thus, the owners of good trees have a choice: sell the tree and pay the tax, generating a utility loss of $\widetilde{\gamma}(\mu, X)\left(\frac{Z}{r_M(X)} - P_M(X)\right)$, or keep the tree and give up the liquidity services associated with it, generating a utility loss of $\left(\frac{\widetilde{\gamma}(\mu,X)}{r_M(X)} - 1\right) Z$. The agents optimally choose the option that generates the smallest loss. In contrast, the pooling price implies an implicit *subsidy* for bad trees. It is the size of this cross-subsidization between good and bad trees, and the *option value* it generates on good trees, that shapes the incentives to produce different qualities.

A consequence of these expressions is that the shadow values have heterogeneous elasticities to market prices. Let $\gamma_i(P_M)$ be the shadow value of type $i \in \{G, B\}$ as a function of future prices $\{P_M(X)\}$, and let $\frac{\partial \gamma_i(P_M)}{\partial P_M(X)}$ be the associated derivative with respect to the market price in state X.[14]

[13] Recall that the marginal utility of consumption in period 2 is equal to 1 for all agents and there is no discounting.

[14] Since prices are a function of the state, the shadow values are *functionals*, so the appropriate concept to measure their change when prices change is the functional derivative. When the space of functions is a Banach space, the corresponding definition is the 'Fréchet" derivative. For an introduction to functional analysis, see Luenberger (1969).

The next proposition presents a key result of the model.

Proposition 2 (Sensitivity of Shadow Values to Prices)**.** *The shadow value of trees satisfy*

$$\frac{\partial \gamma_B(P_M)}{\partial P_M(X)} > \frac{\partial \gamma_G(P_M)}{\partial P_M(X)} \geq 0.$$

Proposition 2 states that the shadow value of bad trees is more sensitive to changes in expected market prices than the shadow value of good trees. Or, put differently, that the private valuation of good trees is more insulated from changes in market conditions than that of bad trees. Good trees have the option value of being kept until maturity if market prices are not sufficiently high, or if liquidity needs are low, while this strategy is always dominated for bad trees. Bad trees are produced only to be sold in the future, that is, for *speculative motives*. Thus, while bad trees are always sold, there are states in which agents strictly prefer not to sell their good trees, insulating their value from price changes. This channel is at the core of the positive and normative analysis that follows.

Finally, I am ready to characterize the agents' choice in period 0. Due to the linearity of the value function in period 1, and using the definition of the shadow value of trees, there is a simple characterization of the agents' optimality conditions.

Lemma 2. *Suppose* $\frac{\gamma_G}{C'(W_0)} < \gamma_B < \frac{\gamma_G}{C'(0)}$. *The agents' production decisions in period* 0 *satisfy*

$$\frac{\gamma_G}{\gamma_B} = C'(H_G) \quad \textit{and} \quad H_B = W_0 - C(H_G). \tag{8}$$

Moreover,

$$\frac{\partial \lambda_E}{\partial P_M(X)} < 0 \quad \textit{and} \quad \frac{\partial H}{\partial P_M(X)} > 0.$$

Given the shadow value of trees, γ_G and γ_B, agents decide which quality of tree to produce by comparing the return per unit invested of each option (good or bad) at the margin. Importantly, the fraction of good trees in the economy, λ_E, is decreasing in market prices, and the total number of trees, H, is increasing in market prices. The effect on λ_E is a corollary of Proposition 2: since the shadow value of bad trees is more sensitive to changes in prices than the shadow value of good trees, the average quality of trees in the economy decreases with market prices. Thus, the production of lemons is more elastic to *future* prices than the production of non-lemons. Moreover, as prices increase and more bad trees are produced, the total number of trees in the economy increases, since bad trees are cheaper to produce than good trees.

Next, I turn to the determination of equilibrium and present the positive analysis of the model.

3 Equilibrium and Market Fragility

In this section, I compute the equilibrium of the economy. First, I characterize the equilibrium in the market for trees for each realization of α, *conditional* on $\{\lambda_E, B_0\}$. Then, I use the characterization of the agents' decisions in period 0 given their expectations about their liquidity needs and the market for trees in period 1, to define a *Rational Expectations Equilibrium*. Finally, I perform a comparative statics analysis to understand the sources of risk build-up in this economy.

3.1 Market for Trees

We can characterize the equilibrium in the market for trees by the *net demand for trees* and a *supply of trees*. The net demand for trees is given by the demand of those agents who have a liquidity shock that is less than $\mu_B(X)$, net of the purchases of government bonds:

$$D(P_M; X) \equiv \frac{G(\mu_B(P_M; X))[W_1 + H_B P_M] - [1 - G(\mu_B(P_M; X))]B_0 Q_1^B(P_M; X)}{P_M}, \tag{9}$$

where $G(\cdot)$ is the cumulative distribution function of μ. Note that buyers sell their bad trees in order to profit from the adverse selection subsidy. Moreover, only agents with $\mu \geq \mu_B(X)$ sell their government bonds, so market clearing implies that only a fraction $1 - G(\mu_B(P_M; X))$ of the total outstanding value of government bonds is traded in the market.

The supply of trees is given by the sum of the good and bad trees in the market, that is,

$$S(P_M; X) \equiv [1 - G(\mu_S(P_M; X))] H_G + H_B. \tag{10}$$

While all agents sell their bad trees, only the fraction of agents with liquidity needs above $\mu_S(P_M; X)$ sell their good trees. Finally, we have that the fraction of good trees in the market is given by

$$\lambda_M(P_M; X) = \frac{[1 - G(\mu_S(P_M; X))] H_G}{S(P_M; X)}. \tag{11}$$

In order to organize the analysis of the equilibrium of the economy, it is useful to define a partial equilibrium of the market for trees in each state X.

Definition 2 (Partial Equilibrium in the Market for Trees)**.** *A partial equilibrium in the market for trees in state X is a price P_M, a fraction of good trees in the market λ_M, and a rate of return r_M, such that the demand for trees* (9) *equals the supply of trees* (10)*, the average quality of trees in the market is given by* (11)*, and*

$$\mu_B(X) = r_M(X).$$

It is well-known that markets that suffer from adverse selection can feature multiple (partial) equilibria. In such cases, the literature typically selects the equilibrium with the highest price, also known as the *maximum volume of trade* equilibrium (see Kurlat, 2013; Chari et al., 2014). Here, I adopt the same convention. The next proposition characterizes the maximum volume of trade equilibrium in this economy.

Proposition 3 (Partial Equilibrium Characterization)**.** *Given a state X, a partial equilibrium always exists. In the unique maximum volume of trade equilibrium, P_M, λ_M and r_M are increasing in α and λ_E. Moreover, there exists $\alpha^*(\lambda_E, B_0) \in [\underline{\alpha}, \overline{\alpha}]$ such that if $\alpha > \alpha^*(\lambda_E, B_0)$, $\lambda_M > 0$ and if $\alpha < \alpha^*(\lambda_E, B_0)$, $\lambda_M = 0$. If $g(\mu^{\max}) > 0$, there exists $\overline{\lambda}_E$ such that if $\lambda_E > \overline{\lambda}_E$, then the maximum volume of trade equilibrium is discontinuous at α^*, that is,*

$$\lim_{\alpha \to \alpha^{*+}} P_M(X) > \lim_{\alpha \to \alpha^{*-}} P_M(X) \quad \textit{and} \quad \lim_{\alpha \to \alpha^{*+}} \lambda_M(X) > \lim_{\alpha \to \alpha^{*-}} \lambda_M(X) = 0.$$

Proposition 3 fully characterizes the market for assets in period 1. First, it shows that, in the maximum volume of trade equilibrium, market prices, the quality of trees in the market, and returns are all increasing in α. When α is high, the adverse selection problem is mild, prices and volume traded in financial markets are high, and the rate of return on assets is high since the liquidity premium is low. In contrast, when α is low, the adverse selection problem is severe, the price of trees is low, and the rate of return on assets is low. In this case, the economy suffers from a high degree of resource misallocation, and bond prices increase, reflecting this problem.

Second, Proposition 3 states the conditions for a discontinuous change in market performance, or a *financial collapse*. Many accounts of the onset of the Great Recession argue that the sudden collapse in the volume traded of mortgage-related assets fits this description. If the supply of trees reacts more strongly to price changes than the demand in the neighborhood of $\lambda_M(X) = 0$, then there is a positive lower bound on the number of good trees in the market in "normal times," that is, absent a financial collapse. This happens when $g(\mu^{\max}) > 0$ (i.e., there is a positive mass of agents with liquidity needs close to the maximum) and λ_E is sufficiently close to 1. However, this also implies that when the demand is too low to be consistent with that level of good trees in the market, the market equilibrium changes discretely to one in which no good trees are traded. This is what happens at α^*. In what follows, I will focus the analysis on economies that satisfy this property. Moreover, note that without additional restrictions on the shape of the demand and supply of trees, the market can feature arbitrary discontinuities in α that are not associated with a financial collapse, i.e., discontinuities that do not lead to $\lambda_M = 0$. To simplify the normative analysis, the following assumption guarantees that the maximum volume of trade equilibrium is continuous in α in normal times.[15]

Assumption 2. *The cumulative distribution function G is weakly convex and weakly log-concave.*

Figure 3 depicts the two scenarios in the space (P_M, λ_M) when Assumption 2 holds.[16] Panel (a) shows a market in which the quality of bad trees is high and the maximum volume of trade

[15]Technically, Assumption 2 guarantees that the supply of tree quality is convex in prices while the demand is concave.

[16]The partial equilibrium is characterized by three equation: (9), (10) and (11). In order to obtain a two-dimensional

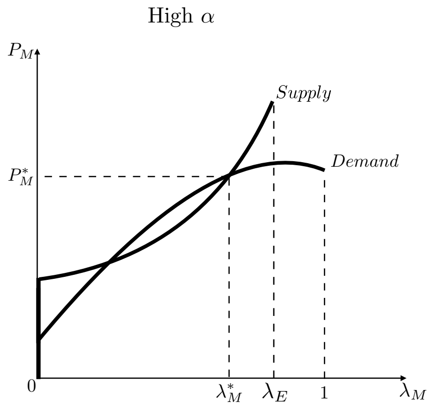

Figure 3: Market Equilibrium in period 1. (a) Multiple Equilibria: Maximum Volume of Trade Selected. (b) Unique Equilibrium: Financial Collapse.

equilibrium features a positive trade of good trees. As the quality of bad trees decreases, the demand moves down. When α is sufficiently low, the economy transitions to the market depicted in Panel (b). In this case, the interior intersection disappears, generating a discontinuous drop in the volume traded.

The previous discussion leads to the following definition of *market fragility*.

Definition 3. *Market fragility is defined as*

$$MF(\lambda_E, B_0) \equiv Prob(\alpha < \alpha^*(\lambda_E, B_0)).$$

Market fragility is the probability of a discontinuous drop in the volume traded in the market for trees. Even though market fragility is not a direct measure of welfare, it is a property that is tightly connected to the efficiency of the economy. A market collapse is an extreme case in which the flow of resources is severely impaired.

3.2 Equilibrium

Next, I define an equilibrium for this economy.

Definition 4. *Given a stock of government debt B_0, a* Rational Expectations Equilibrium *consists of a maximum volume of trade partial equilibrium in the market for trees for every state α, agents' decision rules for the production of trees and consumption, and aggregate variables $\{\lambda_E, H\}$, such that: i) the decision rules solve the agents' problem given the partial equilibria, ii) the partial equilibria are consistent with the agents' decision rules and aggregate variables $\{\lambda_E, H\}$, and iii) $\{\lambda_E, H\}$ are consistent with agents' decision rules.*

representation of the market, I plot a supply and demand for *tree quality* as follows: $Supply: \ \lambda_M = \frac{\left[1-G\left(\frac{Z}{P_M}\right)\right]\lambda_E}{\left[1-G\left(\frac{Z}{P_M}\right)\right]\lambda_E+(1-\lambda_E)}$ and $Demand: \ P_M = \frac{\lambda_M Z+(1-\lambda_M)\alpha Z}{\mu_B(P_M)}$ where $\mu_B(P_M)$ is implicitly defined by the solution to $G(\mu_B)W_1 = \frac{1-\lambda_E}{1-\lambda_M}HP_M - (1-\lambda_E)HG(\mu_B)P_M + [1-G(\mu_B)]\frac{B_0}{\mu_B}$, given λ_M.

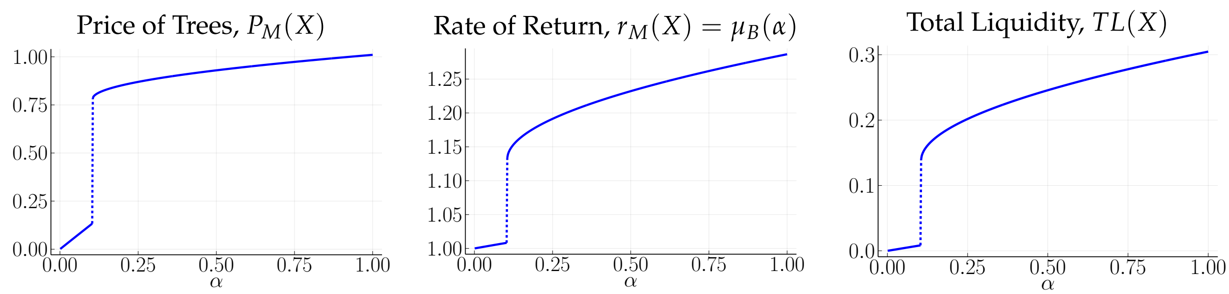

Figure 4: Equilibrium in period 1 as a function of the state α

Note: I assume $\alpha \sim U[0,1]$ and $\mu \sim U[1, \mu^{\max}]$ with $\mu^{\max} = 2$. Moreover, $C(H_G) = \xi \frac{(1+H_G)^{1+\nu}-1}{1+\nu}$, with $\xi = 0.2$ and $\nu = 6$. The other parameters of the model are: $W_0 = 0.25$, $W_1 = W_2 = 1$, and $Z = 1.3$.

In order to complete the characterization of the equilibrium, I have only to determine the fraction of good trees in period 1, λ_E, which is given by

$$\lambda_E = \frac{H_G}{H_G + W - C(H_G)}.$$

Note that the decision to produce trees in period 0 depends on the market prices in period 1. But the prices in period 1 depend on the fraction of good trees in the economy, which is in turn determined in period 0. Let $H_G(\lambda_E)$ denote the aggregate investment in good trees when agents expect the period-2 prices to be consistent with a λ_E fraction of good trees in the economy. Define the following function:

$$T(\lambda_E) = \frac{H_G(\lambda_E)}{H_G(\lambda_E) + W - C(H_G(\lambda_E))}.$$

An equilibrium of this economy requires that $T(\lambda_E) = \lambda_E$.[17] The function T is decreasing in λ_E, since higher λ_E implies higher expected prices, and the result follows from Lemma 2. When the distribution of α is continuous, γ_G and γ_B are continuous functions of λ_E, and hence T is continuous. Therefore, the equilibrium of the economy exists and is unique. The following lemma summarizes these results.

Lemma 3. *A Rational Expectations Equilibrium of the economy always exists and is unique. Moreover,* $\lambda_E \in (0, 1)$.

Figure 4 depicts a numerical example. The equilibrium features $\lambda_E = 0.84$ and $\alpha^* = 0.11$, that is, 84% of the trees in the economy are good and the probability of a crisis is 11%. The figure shows the price of trees, rate of return, and total liquidity, which is defined as

$$TL(X) \equiv \underbrace{Q_B(X)B_0}_{\text{public liquidity}} + \underbrace{[[1 - G(\mu_S(X))]H_G + H_B]\, P_M(X)}_{\text{private liquidity}}. \tag{12}$$

[17]Note that T is a function of the agents' *expectation* of λ_E. A *Rational Expectations Equilibrium* requires that expected and actual λ_E coincide.

That is, total liquidity is the market value of all the assets traded. Note that total liquidity depends on the supply of government bonds and on the performance of private markets. We can see that the price of trees, rate of return, and total liquidity are all monotonically increasing in α. At $\alpha = \alpha^*$, the equilibrium is discontinuous. To the left of α^* the price of trees and total liquidity is low, which drives the rate of return of the economy down due to the increase in the liquidity premium. In particular, total liquidity to the left of α^* is 93.5% lower than to the right. Thus, a small difference in fundamentals can translate into a large change in outcomes.

Discussion of the model's ingredients. The model has three main ingredients: *i*) an asymmetric information problem with respect to the quality of private assets; *ii*) aggregate risk; and *iii*) an endogenously determined liquidity premium. The combination of asymmetric information and aggregate risk introduces an endogenous probability of an abrupt collapse of the financial markets. Moreover, it opens the possibility of studying state-contingent policies. As I show in Section 4, the planner's trade-offs are very different if α is low than if α is high. The endogenous liquidity premium provides a connection between the supply of public liquidity and the incentives to produce private assets, which can be exploited in the optimal policy design. From a technical point of view, the assumption that α has a continuous density is crucial for the existence of an equilibrium. If this were not true, the function T could be discontinuous. In that case, it can be shown that a *sunspot* equilibrium always exists.

Finally, it is worth noting that I adopt a *Walrasian* equilibrium concept for the market of trees, where the equilibrium price equalizes the demand and supply. Wilson (1980) and Stiglitz and Weiss (1981) argue that this might not be a sensible equilibrium concept in markets with adverse selection if buyers can obtain a higher return by offering a higher price and attracting a better tree quality composition. In that case, they suggest a *Buyer* equilibrium concept that allows buyers to submit offers quoting prices and quantities, and ration the excess supply. The main properties of my model are not sensitive to this choice (some of the results would need to be restated in terms of changes in the degree of rationing rather than changes in prices), except for the existence of a discontinuity at α^*. Note, however, that the Buyer equilibrium involves a higher degree of sophistication for buyers than the Walrasian equilibrium, as it requires that they know the relationship between prices and the average quality of trees sold rather than simply the average quality of trees sold at the equilibrium price. A weaker notion of Buyer equilibrium, where there cannot be any *local* profitable deviation from the Walrasian equilibrium, would recover the discontinuity at α^* but could still feature some rationing at higher prices. In what follows I focus on the Walrasian equilibrium to simplify the analysis.

3.3 Comparative Statics

Next, I study some comparative statics that highlight the sources of risk build-ups in this economy. Proposition 4 shows that positive shocks to fundamentals can distort the quality production decisions and increase market fragility. Proposition 5 studies the effects of changes in the supply of government bonds, which is one of the building blocks of the normative analysis in Section 4.

The Quality of Bad Trees

Consider the effects of an anticipated (from period 0's perspective) increase in the expected quality of bad trees (or an expected reduction of *default rates*). In particular, suppose that the distribution of α is indexed by a parameter $\theta : F(\alpha|\theta)$, where a higher θ means a better distribution in the First-Order Stochastic Dominance (FOSD) sense. An increase in θ is equivalent to an increase in prices for all states under the initial distribution. From Lemma 2, we know that the partial equilibrium effect is a reduction in the fraction of good trees in the economy, λ_E, and an increase in the total number of trees, H. While the reduction in λ_E feeds back into the prices, dampening the partial equilibrium result, the overall effect is a decline in the asset quality composition.[18]

Let's turn to the analysis of market fragility. Recall that market fragility is the probability that the quality of bad trees, α, is below the threshold α^*, that is, $MF(\lambda_E) = F(\alpha^*(\lambda_E, B_0)|\theta)$. Differentiating this expression with respect to θ, we get

$$\frac{d\mathrm{MF}}{d\theta} = \underbrace{\frac{\partial F(\alpha^*|\theta)}{\partial \theta}}_{\leq 0} + f(\alpha^*;\theta)\underbrace{\frac{\partial \alpha^*(\lambda_E, B_0)}{\partial \lambda_E}}_{<0}\underbrace{\frac{\partial \lambda_E}{\partial \theta}}_{<0}.$$

For example, suppose that the change in F is concentrated at very high values of α, so that $\frac{\partial F(\alpha^*|\theta)}{\partial \theta} = 0$. Then, the effect of the endogenous adjustment mechanism of the economy dominates, and market fragility *increases*. In contrast, when the fraction of good trees in the economy is exogenously given, as in Eisfeldt (2004) and Kurlat (2013), $\frac{\partial \lambda_E}{\partial \theta} = 0$, and market fragility (weakly) *decreases* after the shock. The next proposition summarizes these results.

Proposition 4 (Increase in Bad Trees' Expected Quality)**.** *Consider an anticipated increase in θ, so that $F(\alpha|\theta)$ increases in the FOSD sense. Then,*

i. the total production of trees, H, increases, and the fraction of good trees in the economy, λ_E, decreases;

ii. market prices in period 1, P_M, decrease in every state;

iii. the threshold α^* increases;

iv. the effect on market fragility is ambiguous.

This is an important result because it states that a "positive" shock can endogenously increase the fragility of the financial markets, in the sense of a higher probability of a market collapse. Thus, it formalizes the idea that positive shocks can set the stage for a financial crisis. Moreover, note that if the change in expectations does not reflect a change in the actual distributions (in the sense that it is just unfounded *optimism*), then fragility always increases after the shock.

[18] There is also an effect on the rate of return which reinforces the price effect. See the proof for the details.

Government Bonds

The previous analysis showed that it is the dual role that trees play that exposes the economy to financial risk. On the one hand, trees are a form of real investment, that is, a technology that transforms goods in one period into goods in other periods. On the other hand, trees facilitate trade in period 1, so that, in the context of incomplete markets, agents can obtain resources even in periods when the trees do not pay any dividend. In reality, the government is an important provider of instruments that perform the second role, particularly through government bonds. Here, I study the channels through which the supply of government bonds can shape the incentives to produce tree quality and affect financial fragility from a positive perspective. In Section 4, I analyze the role of public liquidity from a normative perspective.

Consider the economy in period 1. Suppose that the supply of government bonds in the hands of agents increases exogenously, keeping $\{\lambda_E, H\}$ fixed. The idea is to isolate the market effect in period 1 from the incentives effect in period 0. Recall that the total liquidity in the economy depends on the supply of government bonds and on the performance of private markets. The next lemma shows that the price of trees always decreases with B_0.

Lemma 4. *Consider an economy in period* 1*. Assume that agents' holdings of government bonds increase uniformly, from* B_0 *to* $B_0 + dB_0$*, with* $dB_0 > 0$ *but small, keeping* $\{\lambda_E, H\}$ *fixed. The price of trees and private liquidity decrease in all* α*. There exists* $\tilde{\alpha} > \alpha^*$ *such that for all* $\alpha \in [\alpha^*, \tilde{\alpha})$*, total liquidity decreases. Market fragility increases.*

Keeping $\{\lambda_E, H\}$ fixed, an increase in B_0 reduces the net demand for trees, since government bonds and trees "compete" for the same funds. Thus, an increase in the supply of government bonds *exacerbates* the adverse selection in private markets, and the price of trees decreases. For states close to α^* this effect is sufficiently strong that it generates a private market collapse. Even though the available *public liquidity* increases, if dB_0 is small, the discrete drop in *private liquidity* reduces the total liquidity in the economy, which increases misallocation. Absent any change in the tree quality composition, government bonds increase the fragility in the private markets.

Anticipating the effects of a higher supply of government bonds on the market for trees, agents in period 0 react to higher sales of government bonds by adjusting their quality production.[19] Since the shadow value of bad trees is more sensitive to changes in market conditions than the shadow value of good trees, the quality of trees in the economy unambiguously increases. The next proposition summarizes these results.

Proposition 5 (The Supply of Government Bonds)**.** *Consider an increase in the supply of government bonds in period* 0*. The total production of trees decreases while the production of good trees increases. Thus, the fraction of good trees in the economy,* λ_E*, increases. The effect on market fragility is ambiguous.*

[19]Since all the proceeds from selling bonds in period 0 are rebated to the agents lump-sum, changes in the supply of government bonds affect the production of tree quality only through their effect on the market for trees in period 1.

Proposition 5 formalizes the idea that the scarcity of public safe (or liquid) assets increases the production of private substitutes. But in this model, that production is biased toward low-quality assets. In terms of market fragility, there are two competing forces at play. First, a lower supply of government bonds increases the liquidity premium, which pushes asset prices up. Second, it induces the production of low-quality assets, which depresses the price of private assets. Formally,

$$\frac{dMF}{dB_0} = f(\alpha^*;\theta)\left[\underbrace{\frac{\partial\alpha^*(\lambda_E,B_0)}{\partial B_0}}_{>0} + \underbrace{\frac{\partial\alpha^*(\lambda_E,B_0)}{\partial\lambda_E}}_{<0}\underbrace{\frac{\partial\lambda_E}{\partial B_0}}_{>0}\right].$$

The total effect depends on which effect dominates. If the endogenous production of tree quality is sufficiently responsive to changes in market conditions, i.e. $\frac{\partial\lambda_E}{\partial B_0}$ is sufficiently large, market fragility in the economy increases when government bonds become scarce. In particular, let $\eta_G \equiv \frac{C'(H_G^*)}{C''(H_G^*)}$, where H_G^* is the equilibrium production of good trees. Note that η_G is the semi-elasticity of the production of good trees to changes in shadow values.[20] Then, an increase in η_G increases the responsiveness of the tree quality composition to changes in the supply of government bonds, that is, $\frac{\partial^2\lambda_E}{\partial B_0\partial\eta_G} > 0$.[21]

This results may provide a narrative for some of the developments in the U.S. economy in the years leading to the Great Recession, in which the scarcity of safe assets due to sustained fiscal surpluses in the late 1990s, and the so-called global savings glut in the early 2000s, could have sowed the seeds of the financial crisis, as it put excessive pressure (i.e., generated perverse incentives) on the U.S. financial sector to produce safe assets.[22] This is a period in which the supply of asset quality was probably relatively elastic, as the supply of mortgage-related securities was increasing rapidly. Later on, the public provision of liquidity could have hindered the possibility of restoring the functioning of the private markets, as the production of new assets was low and, therefore, the asset quality composition in the economy was mostly fixed, so the effects of Lemma 4 may have dominated. Of course, this does not imply that the policy was suboptimal. As we shall see, the optimal policy requires an aggressive increase in the supply of public liquidity when restarting the private markets is too costly, but to limit the public provision of liquidity when trying to "jumpstart" the private markets.

4 Welfare and Optimal Policy

In the previous sections, I studied the dynamics of an economy in which market incompleteness and information frictions can lead to a financial crisis. Moreover, I analyzed the positive effects of policy changes, namely, the supply of public liquidity, on equilibrium outcomes. In this section,

[20] To see this, start with $C'(H_G) = \frac{\gamma_G}{\gamma_B}$. Then, $\frac{\partial H_G}{\partial\frac{\gamma_G}{\gamma_B}}\frac{\gamma_G}{\gamma_B} = \frac{C'(H_G)}{C''(H_G)}$.

[21] For a proof of this statement, see the proof of Proposition 5.

[22] See, for instance, Caballero (2006) for a narrative about safe asset shortages.

I explore the model's normative implications by solving the problem of a social planner whose objective is to maximize the expected utility of the representative agent in period 0. The planner faces the same constraints as the private economy; in particular, agents' portfolios and idiosyncratic shock μ are the agents' private information. I begin the analysis by describing a Ramsey problem where the planner has access to two instruments: state-contingent government bonds and transaction subsidies/taxes. I characterize the solution in Sections 4.2 and 4.3. Finally, in Section 4.4 I show that the solution to the Ramsey problem is equivalent to the constrained efficient allocation, in the sense that they induce the same allocation of consumption and production of trees.

4.1 The Ramsey problem

Let a Ramsey plan $\mathcal{P}^R = \{B(\alpha), \omega(\alpha)\}_{\forall\alpha}$ be a set of state-contingent government bonds issued in period 0 and that mature in period 2, where $B(\alpha) \geq 0$ denotes the bond's face value in the aggregate state α, and a set of *market wedges*, $\omega(\alpha)$, which induce the prices $P_S(\alpha)$ and $P_B(\alpha)$ in the market for trees, where $P_S(\alpha)$ denotes the price received by sellers and $P_B(\alpha)$ denotes the price paid by buyers, and $P_S(\alpha) = (1+\omega(\alpha))P_B(\alpha)$. For a given choice of Ramsey plan $\mathcal{P}^R$, the characterization of the equilibrium of the economy is analogous to the one in *laissez-faire* studied in the previous section.[23] To streamline notation and avoid additional clutter, in what follows I omit the dependence of the equilibrium variables on the Ramsey plan $\mathcal{P}^R$. For example, I denote by $\{P_B(\alpha), P_S(\alpha), \lambda_M(\alpha), \mu_B(\alpha)\}$ the partial equilibrium in the market for trees in state α. The reader should note that the equilibrium variables are functions of $\{B(\alpha), \omega(\alpha)\}_{\forall\alpha}$.

The feasibility of the Ramsey plan $\mathcal{P}^R$ requires that the government satisfies a budget constraint in each period. In period 0, the planner sells government bonds to the agents and rebates the proceeds lump-sum, i.e., $\int_{\underline{\alpha}}^{\overline{\alpha}} Q_0^B(\alpha)B(\alpha)dF(\alpha) = T_0$, where $Q_0^B(\alpha)$ denotes the price of a bond that pays 1 in period 2, state α, and T_0 is a lump-sum transfer to the agents. In period 1, the budget constraint is given by

$$\omega(\alpha)P_B(\alpha)S(\alpha) = Q_1^B(\alpha)[B'(\alpha) - B(\alpha)],$$

where $S(\alpha)$ denotes the supply of trees in state α, $B'(\alpha) - B(\alpha)$ denotes the planner's sale (or purchase if negative) of government bonds, and $Q_1^B(\alpha)$ denotes their price in period 1, state α. In period 2, the planner repays the maturing stock of government bonds and can tax the agents' lump sum. To capture the deadweight loss associated with taxation, I assume that there is a cost $\chi > 0$ per unit of tax revenue collected.[24] Thus, the budget constraint in period 2 is given by

$$\mathbb{T}_2(\alpha) = (1+\chi)B'(\alpha) = (1+\chi)\left[B(\alpha) + \mu_B(\alpha)\omega(\alpha)P_B(\alpha)S(\alpha)\right],$$

[23]For a detailed exposition, see Appendix C.

[24]Some cost of intervention is commonly assumed in the literature; see, e.g. Tirole (2012) and Jeanne and Korinek (2020). For an analysis of the case with $\chi = 0$, see Appendix D.

where $\mathbb{T}_2(\alpha)$ denotes the lump-sum tax, and I used that, in equilibrium, $Q_1^B(\alpha) = \frac{1}{\mu_B(\alpha)}$. Moreover, I impose that $\mathbb{T}_2(\alpha) \leq W_2$ to prevent negative consumption in period 2.

One of the main considerations for the planner is how their plan affects the production of tree quality. In an interior solution, production decisions are determined by

$$\frac{\gamma_G}{\gamma_B} = \frac{E\left[\max\{\mu(1+\omega(\alpha))P_B(\alpha), Z\}\right]}{E\left[\max\{\mu, \widetilde{\mu}_B(\alpha)\}(1+\omega(\alpha))P_B(\alpha)\right]} = C'(H_G) \tag{13}$$

$$H_B = W_0 - C(H_G), \tag{14}$$

where $\widetilde{\mu}_B(\alpha) \equiv \max\left\{\mu_B(\alpha), \frac{\alpha Z}{(1+\omega(\alpha))P_B(\alpha)}\right\}$. These expressions are analogous to (8) in Section 2, where P_M is replaced by the price received by sellers, $(1+\omega)P_B$, and $\widetilde{\mu}_B$ replaces μ_B. The new variable $\widetilde{\mu}_B$ reflects the fact that if the planner sets ω sufficiently negative, the adverse selection subsidy on bad trees becomes negative, in which case agents sell their bad trees only when their liquidity needs are sufficiently high (similar to the decision to sell good trees).[25] While the planner cannot observe the production of tree quality, it understands that its choices affect the shadow value of trees and therefore determine the incentives to produce tree quality.

Let $c_1(\mu;\alpha)$ denote the equilibrium consumption level of an agent of type μ in state α. Define

$$\begin{aligned}\boldsymbol{U_1}(\alpha) &\equiv \int_1^{\mu^{\max}} \mu c_1(\mu;\alpha) dG(\mu) \\ &= \int_{\mu_B(\alpha)}^{\mu^{\max}} \mu \left[W_1 + \frac{B(\alpha)}{\mu_B(\alpha)}\right] dG(\mu) + \left[\int_{\widetilde{\mu}_B(\alpha)}^{\mu^{\max}} \mu H_B dG(\mu) + \int_{\mu_S(\alpha)}^{\mu^{\max}} \mu H_G dG(\mu)\right](1+\omega(\alpha))P_B(\alpha).\end{aligned}$$

Then, the Ramsey problem is given by

$$\max_{\{\omega(\alpha),B(\alpha)\}_{\forall\alpha},H_G} E\left[\boldsymbol{U_1}(\alpha) + ZH_G + \alpha Z(W_0 - C(H_G)) - \frac{\chi}{1+\chi}\mathbb{T}_2(\alpha)\right] \tag{PP}$$

subject to (13) and $0 \leq \mathbb{T}_2(\alpha) \leq W_2$.

Next, I study how the different instruments interact with the frictions of the economy, and the trade-offs faced by the planner in the design of the optimal policy.

4.2 Liquidity versus production incentives

In what follows, I focus on solutions where the induced equilibrium features $\lambda_E \in (0,1)$. The next proposition characterizes the necessary conditions for an interior solution.

Proposition 6. *Suppose $\{B(\alpha), \omega(\alpha)\}_{\forall\alpha}$ is a solution to the planner's problem that induces an equilibrium with $\lambda_E \in (0,1)$, and that $0 < \mathbb{T}_2(\alpha) < W_2$ for all α. If $\boldsymbol{U_1}(\alpha)$ and $\mathbb{T}_2(\alpha)$ are differentiable at*

[25] In the *laissez-faire* economy, the adverse selection subsidy on bad trees was always weakly positive.

$\{B(\alpha), \omega(\alpha)\}$*, then, in an interior solution,* $\{B(\alpha), \omega(\alpha)\}$ *satisfy*

$$\underbrace{\left[\frac{\partial \boldsymbol{U}_1(\alpha)}{\partial B(\alpha)} - \frac{\chi}{1+\chi}\frac{\partial \mathbb{T}_2(\alpha)}{\partial B(\alpha)}\right]}_{\text{liquidity effect}} f(\alpha) + \underbrace{E\left[\frac{\partial \boldsymbol{U}_1(\alpha)}{\partial H_G} + (1-\alpha C'(H_G))\, Z - \frac{\chi}{1+\chi}\frac{\partial \mathbb{T}_2(\alpha)}{\partial H_G}\right]\frac{\partial H_G}{\partial B(\alpha)}}_{\text{incentives effect}} = 0 \tag{15}$$

$$\underbrace{\left[\frac{\partial \boldsymbol{U}_1(\alpha)}{\partial \omega(\alpha)} - \frac{\chi}{1+\chi}\frac{\partial \mathbb{T}_2(\alpha)}{\partial \omega(\alpha)}\right]}_{\text{liquidity effect}} f(\alpha) + \underbrace{E\left[\frac{\partial \boldsymbol{U}_1(\alpha)}{\partial H_G} + (1-\alpha C'(H_G))\, Z - \frac{\chi}{1+\chi}\frac{\partial \mathbb{T}_2(\alpha)}{\partial H_G}\right]\frac{\partial H_G}{\partial \omega(\alpha)}}_{\text{incentives effect}} = 0 \tag{16}$$

for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$*, with*

$$\frac{\partial H_G}{\partial B(\alpha)} > 0 > \frac{\partial H_G}{\partial \omega(\alpha)}.$$

Moreover, starting from laissez-faire*, the liquidity benefits per unit spent is higher for market interventions than for public injections, that is*

$$\frac{\frac{\partial \boldsymbol{U}_1(\alpha)}{\partial B(\alpha)}}{\frac{\partial \mathbb{T}_2(\alpha)}{\partial B(\alpha)}} \leq \frac{\frac{\partial \boldsymbol{U}_1(\alpha)}{\partial \omega(\alpha)}}{\frac{\partial \mathbb{T}_2(\alpha)}{\partial \omega(\alpha)}},$$

with strict inequality if good trees are traded.

The necessary optimality conditions for an interior solution have two components: a *liquidity effect* in period 1, and an *incentives effect* in period 0. The liquidity effect captures the impact of the planner's plan on the reallocation of resources for a given composition of tree quality, $\{H_G, H_B\}$. This is the force that justifies government intervention in models in which the private sector fails to fully reallocate resources to the agents with the highest valuations, as in Woodford (1990) and Holmström and Tirole (1998). Notably, Proposition 6 states that starting from *laissez-faire*, the liquidity benefits per unit spent is higher for market interventions, strictly so if good trees are traded.[26] In states where good trees are traded, market interventions have two benefits relative to public injections. First, market interventions induce more agents to sell their good trees, reinforcing the positive effect on market prices and, thus, amplifying the increase in liquidity. Second, market interventions benefit proportionally more the agents with $\mu \in [\mu_S(\alpha), \mu^{\max}]$, so they lead to a better allocation of resources. Thus, from a liquidity perspective, market interventions are a better instrument than public liquidity provision.

Anticipating how the planner's choices affect the functioning of markets in period 1, agents respond by adjusting their production of tree quality in period 0. The planner internalizes these dynamics through the *incentives effect*. The incentives effect reflects how (expected) changes in market liquidity in period 1 affect the determination of the tree quality composition in the economy in period 0. The incentives effect has two components. First, there is the direct change in the

[26]Starting from $B(\alpha) > 0$ or $\omega(\alpha) \neq 0$ introduces the effect that changes in these instruments have on the interest rate, $r_M(\alpha)$. Numerical explorations indicate that the results from Proposition 6 also extend to those cases.

planner's objective for a given change in H_G, which is given by

$$E\left[\frac{\partial \boldsymbol{U}_{\mathbf{1}}(\alpha)}{\partial H_G}+\left(1-\alpha C'(H_G)\right)Z-\frac{\chi}{1+\chi}\frac{\partial \mathbb{T}_2(\alpha)}{\partial H_G}\right]>0.$$

This effect is common to *all* states α and is independent of the instrument that triggered the change. An improvement in the asset quality composition of the economy is always welfare-enhancing.

Second, the incentives effect depends on how the change in a particular instrument affects the determination of H_G. Given a planner's plan, agents choose the quality of trees they produce according to their shadow values (see equation (13)). Consider the incentives effect of a market intervention. An increase in $\omega(\alpha)$ in some state increases the prices in those states, *ceteris paribus*. Anticipating that the price of trees will be higher, the shadow value of bad trees increases by more than that of good trees, leading to an increase in the production of bad trees. Thus, the incentives effect of market interventions is negative. In contrast, the incentives effect of public liquidity is positive. An increase in public liquidity reduces the liquidity premium, which in turn reduces the price of trees. Then, the opposite logic from a market intervention follows: anticipating that the price of trees will be lower in some states, the shadow value of bad trees decreases by more than that of good trees, leading to a reduction in the production of bad trees.

Thus, the planner trades off the relative liquidity benefits of market interventions with the relative incentives benefits of public liquidity. Interestingly, note that, in an interior solution, the liquidity effect of public liquidity is *negative*, implying that the planner over-provides liquidity relative to a pure liquidity motive in order to take advantage of the positive incentives effect.[27]

Finally, note that the expressions in Proposition 6 do not hold at $P_S(\alpha)=\frac{Z}{\mu^{\max}}$ (the threshold for a market collapse). This opens up the possibility of discontinuities in the optimal policy and financial crises as part of the planner's optimal plan.

4.3 Optimal policy

I am ready to characterize the optimal policy. The next proposition summarizes the planner's choice.

Proposition 7 (Optimal Policy)**.** *Suppose that the solution to the planner's problem induces an equilibrium with $\lambda_E \in (0,1)$, $0<\mathbb{T}_2(\alpha)<W_2$ for all α, and that Assumption 2 holds. Then, the optimal policy is characterized by two thresholds $\tilde{\alpha}^*, \tilde{\alpha}^{**}$, with $\underline{\alpha}\leq\tilde{\alpha}^*\leq\tilde{\alpha}^{**}\leq\overline{\alpha}$ such that:*

i. if $\alpha<\tilde{\alpha}^*$, the market for trees collapses, i.e. $\lambda_M(\alpha)=0$, there is a constant transaction tax $\omega(\alpha)<0$, the price received by sellers, $P_S(\alpha)$, is increasing in α, and the level of government bonds, $B(\alpha)$, is decreasing in α;

[27] In Appendix D, I show that the incentives effect can also justify the public provision of liquidity in a model where the agents can borrow from each other, and χ represents the cost of enforcing contracts.

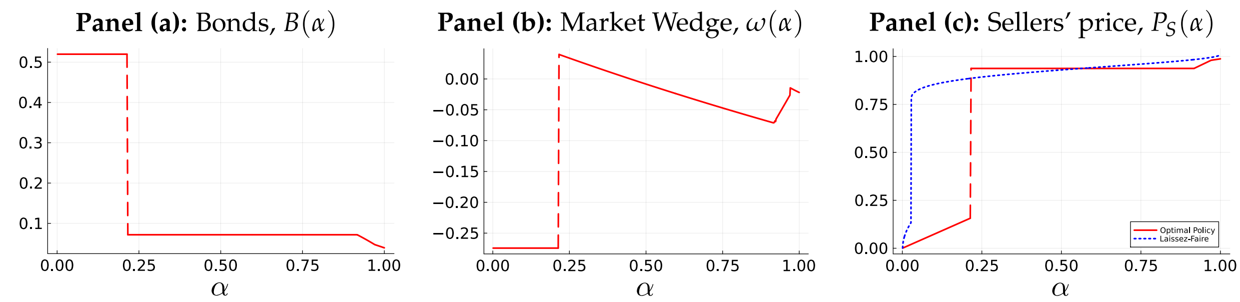

Figure 5: Optimal policy as a function of the aggregate state α

Note: I assume $\alpha \sim U[0,1]$ and $\mu \sim U[1, \mu^{\max}]$ with $\mu^{\max} = 2$. Moreover, $C(H_G) = \xi \frac{(1+H_G)^{1+\nu}-1}{1+\nu}$, with $\xi = 0.2$ and $\nu = 6$. The other parameters of the model are: $W_0 = 0.25$, $W_1 = W_2 = 1$, and $Z = 1.3$. The deadweight loss of transfers is $\chi = 0.17$.

ii. if $\alpha \in [\tilde{\alpha}^*, \tilde{\alpha}^{**})$, then there is positive trade of good trees, i.e. $\lambda_M(\alpha) > 0$, there is a transaction subsidy $\omega(\alpha) > 0$, which is decreasing in α, and the price received by sellers, $P_S(\alpha)$, and the level of government bonds, $B(\alpha)$, are constant in α;

iii. if $\alpha \in (\tilde{\alpha}^{**}, \overline{\alpha}]$, then there is positive trade of good trees, i.e. $\lambda_M(\alpha) > 0$, there is transaction tax $\omega(\alpha) < 0$, the price received by sellers, $P_S(\alpha)$, is weakly increasing in α, and the level of government debt, $B(\alpha)$, is weakly decreasing in α.

Moreover, if $\tilde{\alpha}^* > \underline{\alpha}$, then

$$\lim_{\alpha \to \tilde{\alpha}^{*-}} P_S(\alpha) < \lim_{\alpha \to \tilde{\alpha}^{*+}} P_S(\alpha) \quad \textit{and} \quad \lim_{\alpha \to \tilde{\alpha}^{*-}} B(\alpha) > \lim_{\alpha \to \tilde{\alpha}^{*+}} B(\alpha).$$

Proposition 7 completely characterizes the optimal policy. Figure 5 depicts a numerical example of the optimal level of government bonds, $B(\alpha)$, the optimal market wedge, $\omega(\alpha)$, and the corresponding price for sellers, $P_S(\alpha)$, as functions of the aggregate state α. In this example, the optimal policy induces an equilibrium with $\lambda_E = 0.9$, while the *laissez-faire* equilibrium has $\lambda_E = 0.84$. That is, the optimal policy leads to an increase in the asset quality produced.

The optimal policy distinguishes between three regions of intervention characterized by the level of α: *i*) a region of private market collapse and high public liquidity; *ii*) a region of support of the private market with a positive market wedge and low public liquidity; and *iii*) a region with high prices, negative market wedge, and low public liquidity. Thus, we can identify two main characteristics of the optimal policy: aggressive direct provision of liquidity in *crisis* states, and *leaning against the wind* in *normal* states. This profile reflects the properties of the instruments discussed above.

When α is low, the optimal policy induces a financial crisis, in which case only bad trees are traded. The planner allows the market to collapse as it would be too costly in terms of incentives to support the market when bad trees have their worst performance. However, the planner does not allow the *total liquidity* in the economy to collapse, as it partially substitutes private liquidity with public liquidity. Figure 6 Panel (a) shows the total liquidity in the economy as a function of α. We see that the planner targets a relatively stable level of total liquidity across states (note that total liquidity is *higher* in the crisis states; I will come back to this below). Interestingly, in the region of a private market collapse, the planner introduces a negative market wedge, such that $P_S(\alpha) < P_B(\alpha)$, reducing the private liquidity even further (Figure 5 Panel (c)). To understand this result, note that, in a crisis state, the market for trees does not suffer from adverse selection. This implies that the planner can implement a small tax on trees and use the proceeds to increase the provision of public liquidity, which effectively keeps the total liquidity in the economy unchanged. However, by reducing the price received by sellers, the incentives to produce bad trees decrease, increasing overall welfare. That is, crisis states are a good time to tax a market where only bad trees are traded.[28]

At $\alpha = \tilde{\alpha}^*$, the planner changes its policy discontinuously: it reduces the direct provision of liquidity and increases the market wedge. The positive market wedge increases the price received by sellers, which induces them to sell more good trees, increasing the liquidity in the private market. Moreover, by reducing public liquidity, the planner induces an increase in the liquidity premium, which increases the price of trees for a given market wedge. Thus, both policies *increase* the price of trees and, therefore, contribute to stimulating the private market. Note, however, that in this numerical exercise, the planner chooses a probability of a private market collapse that is higher than the one in the *laissez-faire* equilibrium. Yet, Figure 6 Panel (a) shows that the planner does not allow the *total liquidity* in the economy to collapse. That is, the planner's goal is to avoid the severe resource misallocation associated with financial crises rather than the collapse of private markets *per-se*. Naturally, the cost of taxation, χ, is crucial for this result. As χ increases, the cost of transfers increases, so compensating for a private market collapse with public liquidity becomes more costly. Consequently, the planner chooses a plan that relies less on public liquidity provision and more on supporting the private market, which effectively *reduces* the probability of a financial crisis.[29] Thus, in the optimal policy, a *lower* cost of intervention can be associated with a *higher* probability of a financial crisis, though its consequences are not as severe as in the *laissez-faire* equilibrium.[30] In the numerical example, a 1% reduction in χ generates an increase in the probability of a financial crisis of 60 basis points.

For higher values of α, the planner targets a constant level of prices and public liquidity (except for the highest levels). Moreover, as noted before, total liquidity is lower in these states than in the crisis states. However, this is not reflected in $\boldsymbol{U_1}(\alpha)$, which is higher in normal states (Figure 6 Panel (b)). It might seem surprising that $\boldsymbol{U_1}(\alpha)$ is higher even though the total liquidity is lower. The reason for this result is that the private market transfers relatively more resources to high-μ agents (recall that $\mu_S(\alpha) > \mu_B(\alpha)$). Thus, to achieve the same level of utility as private markets, the

[28] Because the volume traded is low, the revenue collected is relatively low.

[29] However, the economy ends up with a higher fraction of bad trees.

[30] Absent the incentives effect, the optimal policy would only provide market support and the probability of a financial crisis would be lower than in the *laissez-faire* equilibrium (see Tirole, 2012). It is the endogenous production of tree quality that makes the public provision of liquidity beneficial.

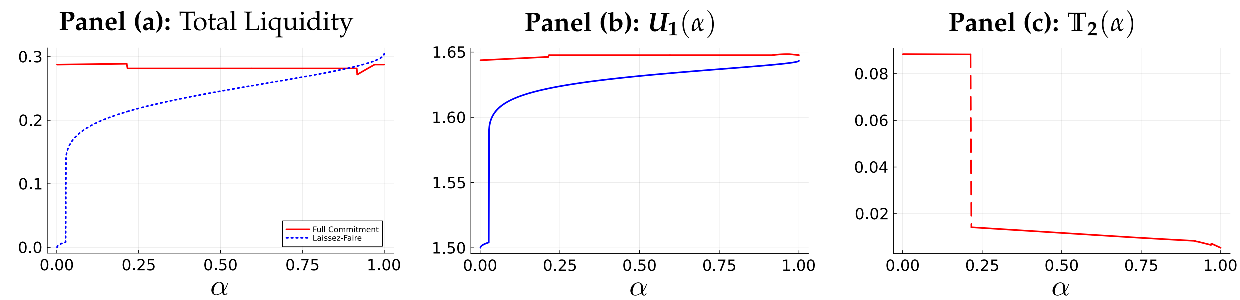

Figure 6: Optimal policy outcomes as a function of the aggregate state α

Note: I assume $\alpha \sim U[0,1]$ and $\mu \sim U[1, \mu^{\max}]$ with $\mu^{\max} = 2$. Moreover, $C(H_G) = \xi \frac{(1+H_G)^{1+\nu}-1}{1+\nu}$, with $\xi = 0.2$ and $\nu = 6$. The other parameters of the model are: $W_0 = 0.25$, $W_1 = W_2 = 1$, and $Z = 1.3$. The deadweight loss of transfers is $\chi = 0.17$.

planner would need to provide a higher level of public liquidity. But, as Figure 6 Panel (c) shows, the direct provision of liquidity is significantly more costly than supporting the private market. In fact, in the numerical example, the difference in the deadweight loss around $\tilde{\alpha}^*$ is orders of magnitude larger than the difference in $\boldsymbol{U}_1(\alpha)$, showing that the choice of $\tilde{\alpha}^*$ is driven by a trade-off between the cost of direct liquidity provision and the incentives to produce tree quality that the policies generate, rather than about the amount of liquidity provided in the market, which is the main force in the policy analysis of Tirole (2012).

Finally, when α is sufficiently high, such that good and bad trees have similar payoffs, the planner chooses a negative market wedge but allows prices to increase in α, reducing the prevalence of public liquidity. Since good and bad trees are similar in these states, the incentives benefit of high taxes are relatively small. Thus, while these states *always* feature a negative market wedge, the wedge's magnitude can be increasing in α (i.e., less negative). This implies that while a *leaning against the wind* type of policy is optimal in the high-α states, the strength of the leaning might be non-monotonic.

Asset purchase programs. The previous analysis implied that a positive market wedge, i.e. transaction subsidies, is a crucial component of the optimal policy. However, transaction subsidies might be problematic, as they are likely to generate spurious trades exclusively aimed at collecting the subsidy.[31] Thus, it would be useful to have an alternative instrument that could be mapped to the positive market wedge, but that does not suffer from this problem. The next proposition shows that, in a market in which good trees are traded, i.e., $\lambda_M(\alpha) > 0$, asset purchase programs are equivalent to a positive market wedge.[32]

[31] This was ruled out in the previous analysis by assuming that each tree could be traded only once per period. In reality, a large number of trades could happen in a short period of time.

[32] The main properties of the optimal asset purchase program were developed in Tirole (2012). However, the equivalence result is new.

Proposition 8 (Instrument Equivalence)**.** *Consider an economy in period* 1 *with* $\lambda_M(\alpha) > 0$*,* $\omega(\alpha) > 0$ *and* $P_S^{\omega}(\alpha) \equiv (1+\omega(\alpha))P_B(\alpha) < \frac{Z}{\mu_B(\alpha)}$*. There exists an alternative economy with* $\omega(\alpha) = 0$ *and where the government implements an asset purchase program by which it purchases at least* $S_B(\alpha) > 0$ *units of bad trees at a price* $P_S^{\omega}(\alpha)$*, that induces the same equilibrium in period* 1 *and generates the same deadweight loss from transfers. Both economies generate the same incentives in period* 0*.*

The proof of Proposition 8 is divided into two parts. First, I show that there exists $S_B(\alpha) > 0$ such that if the planner purchases $S_B(\alpha)$ units of bad trees, the equilibrium of the economy coincides with that of an economy with a positive wedge. This can be achieved by showing that: *i*) choosing $S_B(\alpha)$ appropriately, the equilibrium price of trees in the economy with the asset purchase program coincides with the price of trees in the economy with subsidies, and *ii*) the deadweight loss in period 2 associated with these policies is the same in both cases.

Let $P_S^{\omega}(\alpha)$ denote the price sellers face in the economy with a transaction subsidy. By setting

$$S_B(\alpha) = H_B - \frac{Z - \mu_B(\alpha) P_S^{\omega}(\alpha)}{\mu_B(\alpha) P_S^{\omega}(\alpha) - \alpha Z}\left[1 - G\left(\frac{Z}{P_S^{\omega}(\alpha)}\right)\right] H_G,$$

we get the same price of trees in the economy with the asset purchase program.[33] Then, after some algebra, we can show that the cost of the asset purchase program is given by

$$\mu_B(\alpha) P_S^{\omega} S_B(\alpha) = \mu_B(\alpha) P_S^{\omega} S(\alpha) - \underbrace{[\lambda_M(\alpha) Z + (1-\lambda_M(\alpha))\alpha Z]}_{=\mu_B(\alpha)P_B(\alpha)} S(\alpha),$$

which is the same as the cost of the transaction subsidy. Intuitively, both policies operate by distorting the adverse selection tax on good trees.

The second part of the proof shows that the asset purchase program can generate other equilibria, but all equilibria are equivalent in terms of the allocations they generate. A key insight is that if the planner announces that it will buy at least $S_B(\alpha)$ units of trees at a price $P_S^{\omega}(\alpha)$, at least $S_B(\alpha)$ of those trees will be bad. Since no-arbitrage implies that $P_S^{\omega}(\alpha)$ is also the price at which trees are traded in the private market, and since $P_S^{\omega}(\alpha)$ is higher than the *laissez-faire* price, no-arbitrage *requires* that the planner purchases at least $S_B(\alpha)$ units of bad trees to sustain the price in the private market. Put differently, if the planner bought less than $S_B(\alpha)$ units of bad trees, then the price in the market would be less than $P_S^{\omega}(\alpha)$, so all agents would sell their trees to the planner, contradicting that the planner purchased less than $S_B(\alpha)$ of bad trees. Still, the planner may end up purchasing more than $S_B(\alpha)$, inducing different equilibria. However, all the equilibria feature the same $\lambda_M(\alpha)$ and, therefore, the same allocation of consumption. One way to interpret these equilibria is as follows. The government purchases a minimum of $S_B(\alpha)$ units of bad trees. Then, it may buy more trees, and the proportion of qualities in the additional purchases is $\lambda_M(\alpha)$ of good trees and $1-\lambda_M(\alpha)$ of bad trees. Because the additional trees are purchased at a fair price,

[33] If $\omega(\alpha) > 0$ and $P_s^{\omega}(\alpha) < \frac{Z}{\mu_B(\alpha)}$, then $S_B(\alpha) > 0$.

they entail no additional cost for the planner and do not affect the agents' total income from the sale of trees (to other agents and to the planner). Then, they all induce the same allocation of consumption.

4.4 An Equivalent Constrained Efficiency Problem

In this section, I have solved a Ramsey problem where the planner is endowed with state-contingent government bonds and transaction taxes/subsidies. I now show that the solution to the Ramsey solution is equivalent to the constrained efficient allocation of a planner that faces information frictions consistent with those faced by the agents of the economy of Section 2. One thing to note, however, is that in the economy of Section 2, the planner would be able to easily infer the agents' portfolios, as there is no heterogeneity in their holdings. To prevent this from happening, I make two additional assumptions. First, in period 0, agents choose how much to invest in good and bad trees, $C(H_G)$ and H_B, but the amount they receive in period 1 is given by $h_j = \zeta H_j$ for $j \in \{G, B\}$ where ζ is a positive random variable with $E[\zeta] = 1$. Second, I assume that the agents' endowment in period 1 is a random variable ω_1 i.i.d. across agents, with $E[\omega_1] = W_1$ and support $[0, \infty)$. Because of the linearity of the agents' problem, these assumptions have no impact on the competitive equilibrium but prevent the planner from trivially inferring the individual agents' portfolios. Since agents' types are their private information, I focus on direct mechanisms where the allocation is conditioned on the agents' announcement of their type, which is given by their idiosyncratic shock, μ, their endowment in period 1, ω_1, and their holdings of trees in period 1, (h_G, h_B).[34] To reduce clutter, let $x = (\mu, \omega_1, h_G, h_B)$. The variable x summarizes the agents' private information.

The planner can control agents' consumption and reallocate tree holdings. Formally, the planner chooses a plan $\mathcal{P} = \left\{c_1\left(x;\alpha\right), c_2\left(x;\alpha\right), h'_G\left(x;\alpha\right), h'_B\left(x;\alpha\right)\right\}_{\forall x,\alpha}$ in order to maximize the agents' expected utility in period 0, where $c_1(x;\alpha)$ and $c_2(x;\alpha)$ denote the consumption of an agent type x in periods 1 and 2, respectively; and $h'_G(x;\alpha)$ and $h'_B(x;\alpha)$ denote their holdings of good and bad trees in period 2, respectively.

The planner faces technological and information constraints. First, a plan $\mathcal{P}$ is *feasible* if it satisfies a resource constraint in each period. In period 1, the resource constraint for goods is given by

$$\int_x c_1(x;\alpha) d\Gamma(x) = W_1,$$

where $c_1(x;\alpha) \geq 0$, and $\Gamma(x)$ denotes the joint distribution of idiosyncratic shock μ, endowment ω_1, and portfolios (h_G, h_B). For period 2, I introduce a cost of redistribution. Let

$$T_2(x;\alpha) \equiv c_2(x;\alpha) - W_2 - Zh'_G(x;\alpha) - \alpha Z h'_B(x;\alpha).$$

[34] In this section I assume that the production of trees is unobservable to the planner. In Appendix E, I discuss a planner than can choose the *total* production of trees in period 0 but not the quality produced.

That is, $T_2(x;\alpha)$ is the difference between the consumption in period 2 of an agent type x and its sources of income, i.e., the sum of the endowment and the proceeds from the portfolio of trees. In other words, $T_2(x;\alpha)$ can be interpreted as transfers to the agents in period 2. I assume that $T_2(x;\alpha)$ can be expressed as $T_2(x;\alpha) = \mathcal{T}_2(x;\alpha) - \mathbb{T}_2(\alpha)$, where $\mathcal{T}_2(x;\alpha) \geq 0$, and the resource constraint satisfies

$$\mathbb{T}_2(\alpha) = (1+\chi)\int_x \mathcal{T}_2(x;\alpha)d\Gamma(x),$$

where $\chi > 0$ represents a deadweight loss from transfers. This specification implies that the planner cannot impose negative transfers in a targeted way, and untargeted negative transfers carry a deadweight loss χ. Moreover, I impose that $\mathbb{T}_2(\alpha) \leq W_2$ to prevent negative consumption in period 2.

The reallocation of trees also needs to be feasible. Let $s_G(x;\alpha)$ and $s_B(x;\alpha)$ denote the number of good and bad trees, respectively, that a type-x agent transfers to the planner. Similarly, let $m_G(x;\alpha)$ and $m_B(x;\alpha)$ denote the number of good and bad trees, respectively, received by a type-x agent. Then, the law of motion of the agents' tree holdings satisfies

$$h'_j(x;\alpha) = h_j - s_j(x;\alpha) + m_j(x;\alpha), \quad j \in \{G, B\}.$$

Moreover, the planner faces information constraints. A feasible plan is *incentive compatible* if it satisfies the following incentive compatibility constraints:

$$\mu c_1(x;\alpha) + c_2(x;\alpha) \geq \mu c_1(x';\alpha) + c_2(x';\alpha), \quad \forall x, x' \in \mathbb{X}. \tag{17}$$

Additionally, I make the following assumption.

Assumption 3. *Let $\Omega(x;\alpha) \equiv c_1(\mu,\omega_1,h_G,h_B;\alpha) - c_1(\mu,\omega_1,0,0;\alpha)$.*

i. For every $n \in \mathbb{N}$ and every $\{h_j^i\}_{i=1}^n$ with $h_j^i \geq 0$ and $\sum_{i=1}^n h_j^i = h_j$ for $j \in \{G, B\}$,

$$\Omega(\mu,\omega_1,h_G,h_B;\alpha) = \sum_{i=1}^n \Omega(\mu,\omega_1,h_G^i,h_B^i;\alpha).$$

ii. For every ω_1,

$$c_1(\mu,\omega_1,h_G,h_B;\alpha) = c_1(\mu,\omega_1 + \Omega(\mu,\omega_1,h_G,h_B;\alpha),0,0;\alpha).$$

Assumption 3 captures the essence of anonymity and non-exclusivity assumed for the private markets. The assumption implies that the planner cannot individually identify the agents by their portfolios. Note that the variable $\Omega(x;\alpha)$ can be interpreted as the reward for holding a portfolio (h_G, h_B) of trees. Assumption *i*. states that the planner cannot determine if an announcement (h_G, h_B) came from one agent holding all the trees or many agents each holding (h_G^i, h_B^i) with $\sum_{i=1}^n h_j^i = h_j$ for $j \in \{G, B\}$. Thus, the planner cannot distinguish the agents by the size of their portfolios. This implies that $c_1(x;\alpha)$ will be piecewise linear in (h_G, h_B).[35] Assumption *ii*. states that the planner cannot condition the consumption allocation on the agents' source of "income" (the endowment or the reward from holding trees). For example, the agents can hide their rewards from trees and claim that their endowment is $\hat{\omega}_1 = \omega_1 + \Omega(\mu, \omega_1, h_G, h_B; \alpha)$ instead of ω_1. This implies that the planner cannot condition the terms of trades for the reallocation of trees on the consumption allocation.[36]

Finally, agents make production decisions in period 0 according to the incentives induced by the plan $\mathcal{P}$. In particular, agents' production decisions are the solution to

$$\max_{(H_G,H_B)\geq 0} \mathbb{E}_0[\mu c_1(x;\alpha) + c_2(x;\alpha)] \quad s.t. \quad C(H_G) + H_B = W_0,$$

where $x = (\mu, \omega_1, \xi H_G, \xi H_B)$. Since $\mathbb{E}[\xi] = 1$, the solution to this problem determines the *aggregate* level of production (H_G, H_B), and the individual levels are then given by $h_G = \xi H_G$ and $h_B = \xi H_B$.

Let $\mathbb{FIP}$ denote the set of feasible and incentive-compatible plans that satisfy Assumption 3. The optimal plan is a plan in $\mathbb{FIP}$ that maximizes the agents' expected utility in period 0 subject to their production decisions; that is, it solves

$$\mathbb{W} \equiv \max_{\mathcal{P}\in\mathbb{FIP}} \mathbb{E}_0\left[\mu c_1(x;\alpha) + c_2(x;\alpha)\right] \tag{PP'}$$

subject to

$$(H_G, H_B) \in \arg\max_{(\widetilde{H}_G,\widetilde{H}_B)\geq 0} \mathbb{E}_0[\mu c_1(x;\alpha) + c_2(x;\alpha)] \quad s.t. \quad C(\widetilde{H}_G) + \widetilde{H}_B = W_0.$$

I am ready to present the equivalence result between the constrained efficient planner's problem and the Ramsey problem.

Proposition 9 (Equivalence)**.** *The planner's problem* (PP') *is equivalent to the Ramsey problem* (PP)*, in the sense that their solutions induce the same allocation of consumption and production of trees.*

The intuition for the proof of Proposition 9 is as follows. Since agents' idiosyncratic shock μ is unobservable, and the planner wants to transfer consumption goods to high-μ agents in period 1, truthful revelation requires that the planner compensates low-μ agents with consumption in period 2. It can achieve this in two ways: it can promise direct transfers, or it can reallocate trees and let the agents consume the trees' payoffs. Moreover, since the planner cannot see portfolios, any transfer conditional on tree holdings must involve the transfer of trees to the planner to make

[35] The linearity assumption is widespread in the literature on optimal contracts because of its realism and tractability. See Bolton and Dewatripont (2004), Chapter 4 for a discussion.

[36] The anonymity of private markets implied that market participants could not see other agents' trading and consumption decisions. Otherwise, this information would reveal the quality of the trees they were selling. Assumption 3 puts the agents and the planner on equal grounds in terms of the information they can use to identify agents' types.

it incentive compatible. Thus, consumption in period 1 can be written as

$$c_1(x;\alpha) = T_1(\mu, \omega_1; \alpha) + \overline{P}_S(x;\alpha)(s_G(x;\alpha) + s_B(x;\alpha)),$$

where $T_1(\mu, \omega_1; \alpha)$ is a transfer conditional on the liquidity shock μ and the endowment ω_1, and $\overline{P}_S(x;\alpha)$ denotes the *average* compensation for transferring s_G and s_B to the planner for an agent type x. To get the equivalence with the Ramsey problem, we need three additional observations. First, the planner allocates positive transfers in period 1 only to agents with $\mu \geq \mu_B(\alpha)$ for some $\mu_B(\alpha) \in [1, \mu^{\max}]$, and these transfers are independent of μ; in particular, $T_1(\mu, \omega_1; \alpha) = \omega_1 + \overline{T}_1(\alpha)$, with $\overline{T}_1(\alpha) \geq 0$. Second, Assumption 3 implies that $c_1(x;\alpha)$ is linear in (h_G, h_B) and independent of μ if $\mu \geq \mu_B(\alpha)$, so that $\overline{P}_S(x;\alpha) = P_S(\alpha)$ for $\mu \geq \mu_B(\alpha)$ (and $P_S(x;\alpha) = 0$ for $\mu < \mu_B(\alpha)$). Moreover, if $h_B > 0$ but $\mu < \mu_B(\alpha)$, the planner offers the agents $\mu_B(\alpha)P_S(\alpha)h_B$ in period 2. This is a consequence of Assumption 3 *ii.*, which forces the planner to treat the proceeds from transferring trees, $P_S(\alpha)h_B$, and the agents' endowment, ω_1, analogously. Third, the reallocation of trees is such that $m_G(x;\alpha) = \lambda_M(\alpha)m(x;\alpha)$ and $m_B(x;\alpha) = (1 - \lambda_M(\alpha))m(x;\alpha)$, with $m(x;\alpha) > 0$ only if $\mu < \mu_B(\alpha)$, and the benefit of receiving a tree from period 1's perspective is $P_B(\alpha) = \frac{\lambda_M(\alpha)Z + (1-\lambda_M(\alpha))\alpha Z}{\mu_B(\alpha)}$. Thus, setting $\omega(\alpha) = \frac{P_S(\alpha)}{P_B(\alpha)} - 1$ and $B(\alpha) = \mu_B(\alpha)\overline{T}_1(\alpha)$, we get the desired result.

5 Conclusion

I have presented a model in which the *ex-ante* production of assets interacts with the *ex-post* adverse selection in financial markets, exposing the economy to episodes that feature a sudden collapse in the volume traded in private markets, i.e., a *financial crisis*. Assets in the economy derive value from the dividend they pay and the liquidity services they provide. As a consequence, the supplies of privately produced assets and government bonds (i.e., private and public liquidity) interact through an endogenously determined liquidity premium.

This model provides a useful laboratory to study the optimal policy *mix*, where the planner can directly intervene in the private markets or actively manage the amount of public liquidity. I showed that the optimal policy can be implemented with three instruments: state-contingent government bonds, asset purchase programs, and transaction (or *Tobin*) taxes. Notably, the optimal policy does not rule out the possibility of a financial crisis, but it aggressively increases the supply of public liquidity in such an event, in order to mitigate sharp variations in the total amount of liquidity. In contrast, when the optimal policy prescribes supporting the private market, the planner chooses a combination of an asset purchase program and a high liquidity premium (by reducing the stock of government bonds), which boosts asset prices. Moreover, the planner finds it optimal to implement a transaction tax in the states with the lowest and highest levels of α.

An essential feature of the solution is the need for a state-contingent provision of public liquidity, which I model as state-contingent government bonds. An interesting alternative would be to determine whether a state-contingent monetary policy that manages the market value of non-contingent bonds can play the same role. I conjecture that *conventional* policy alone would not be sufficient, but policies like "Operation Twist," which affects the *composition* of government debt, might provide the additional tool necessary for the implementation. Studying this dimension of the problem would require building a model with government debt of multiple maturities and frictions that make these different assets imperfect substitutes. I leave this question for future research.

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A Proofs

Proof of Proposition 1. The first-order conditions associated to the first best program are:

$$
\begin{aligned}
(c_1(\mu,\alpha)):&\quad \mu \le \gamma_1(\alpha)\\
(c_2(\mu,\alpha)):&\quad 1 \le \gamma_2(\alpha)\\
(H_G):&\quad E[Z\gamma_2(\alpha)] \le \gamma_0 C'(H_G)\\
(H_B):&\quad E[\alpha Z\gamma_2(\alpha)] \le \gamma_0
\end{aligned}
$$

where γ_0, $\gamma_1(\alpha)$ and $\gamma_2(\alpha)$ are the Lagrange multipliers associated to the resource constraint in periods 0, 1 and 2, respectively. Thus, $\gamma_0 = \frac{Z}{C'(C^{-1}(W_0))} > E[\alpha Z]$, $\gamma_1(\alpha) = \mu^{\max}$, $\gamma_2(\alpha) = 1$, where the inequality holds by Assumption 1. Then, in the first best allocation only good trees are produced and agents with $\mu = \mu^{\max}$ consume all the endowment in period 1. Any allocation of the consumption good in period 2 is consistent with first best.

□

Proof of Lemma 1. Consumption in period 1 of an agent with a portfolio (h_G, h_B, b) and idiosyncratic shock μ, is given by

$$
c_1 = \begin{cases} 0 & \text{if } \mu < \mu_B(X) \\ W_1 + h_B P_M(X) + bQ_1^B(X) & \text{if } \mu \in [\mu_B(X), \mu_S(X)) \\ W_1 + (h_B + h_G)P_M(X) + bQ_1^B(X) & \text{if } \mu \ge \mu_S(X), \end{cases}
$$

and in period 2

$$
c_2 = \begin{cases} \mu_B(X)(W_1 + h_B P_M(X)) + Zh_G + b + W_2 + T_2(X) & \text{if } \mu < \mu_B(X) \\ Zh_G + W_2 + T_2(X) & \text{if } \mu \in [\mu_B(X), \mu_S(X)) \\ W_2 + T_2(X) & \text{if } \mu \ge \mu_S(X), \end{cases}
$$

where I used that agents always sell their bad trees and that they receive a return $r_M(X) = \mu_B(X)$ for the liquid wealth they save. Using that $Q_1^B(X) = \frac{1}{\mu_B(X)}$ and that $V_1(h_G, h_B, b; \mu, X) = \mu c_1 + c_2$, we get the desired result.

□

Proof of Proposition 2. Using that $\mu_B(X)P_M(X) = \lambda_M(X)Z + (1 - \lambda_M(X))\alpha Z$, we have $\frac{\partial \gamma_G(P_M)}{\partial P_M(X)} = \int_{\mu_S(X)}^{\mu^{\max}} \mu dG(\mu) f(\alpha)$, $\frac{\partial \gamma_B(P_M)}{\partial P_M(X)} = \left[G(\mu_B(X))(1-\alpha)Z\frac{\partial \lambda_M(X)}{\partial P_M(X)} + \int_{\mu_B(X)}^{\mu^{\max}} \mu dG(\mu)\right] f(\alpha)$, where $\frac{\partial \lambda_M(X)}{\partial P_M(X)} = \frac{\lambda_M(X)(1-\lambda_M(X))}{P_M(X)} \frac{g\left(\frac{Z}{P_M(X)}\right)\frac{Z}{P_M(X)}}{1-G\left(\frac{Z}{P_M(X)}\right)} > 0$. Since $\mu_S(X) \ge \mu_B(X)$, $\frac{\partial \gamma_B(P_M)}{\partial P_M(X)} > \frac{\partial \gamma_G(P_M)}{\partial P_M(X)} \ge 0$.

□

Proof of Lemma 2. Let $\gamma_j \equiv E[\widetilde{\gamma}_j(\mu, X)]$ for $j \in \{G, B, GB\}$. Then, the first-order conditions associated to the agent's problem in period 0 are given by The first-order conditions are given by

$$
\begin{gathered}
\gamma_G - \kappa C'(h_G) \le 0\\
\gamma_B - \kappa \le 0\\
\gamma_{GB} - \kappa Q_0^B \le 0
\end{gathered}
$$

where κ is the Lagrange multiplier associated with the budget constraint. Assuming that $\frac{\gamma_G}{C'(W_0)} < \gamma_B < \frac{\gamma_G}{C'(0)}$, we get $\frac{\gamma_G}{\gamma_B} = C'(H_G)$. Moreover, since $T_0 = Q_0^B B$, we have $H_B = W - C(H_G)$.

Next, note that, if $H_B > 0$, we have $\frac{\partial H}{\partial P_M(X)} = \frac{\partial H_G}{\partial P_M(X)} + \frac{\partial H_B}{\partial P_M(X)}$, where $\frac{\partial H_B}{\partial P_M(X)} = -C'(H_G)\frac{\partial H_G}{\partial P_M(X)}$, and $\frac{\partial H_G}{\partial P_M(X)} = \frac{\frac{\partial \gamma_G}{\partial P_M(X)}\frac{1}{\gamma_G} - \frac{\partial \gamma_B}{\partial P_M(X)}\frac{1}{\gamma_B}}{\frac{C''(H_G)}{C'(H_G)}}$. Since $\frac{\partial \gamma_G}{\partial P_M(X)}\frac{1}{\gamma_G} - \frac{\partial \gamma_B}{\partial P_M(X)}\frac{1}{\gamma_B} < 0$, $\frac{\partial H_G}{\partial P_M(X)} < 0$. Moreover, since $C'(H_G) = \frac{\gamma_G}{\gamma_B} > 1$, $\frac{\partial H}{\partial P_M(X)} > 0$. Finally, $\frac{\partial \lambda_E}{\partial P_M(X)} \propto \frac{\partial H_G}{\partial P_M(X)} - \lambda_E \frac{\partial H}{\partial P_M(X)} < 0$.

□

Proof of Proposition 3. It will be useful to work with a market for *tree quality*. Define the *supply of tree quality as* $\lambda_M^S = \frac{\left[1-G\left(\frac{Z}{P_M}\right)\right]H_G}{\left[1-G\left(\frac{Z}{P_M}\right)\right]H_G + H_B}$, and the *demand for tree quality* as $\lambda_M^D = \frac{\mu_B(P_M)P_M - \alpha Z}{(1-\alpha)Z}$, where $\mu_B(P_M)$ is the implicit function defined as the solution to $G(\mu_B) W_1 = \frac{1}{1-\lambda_M} H_B P_M - H_B G(\mu_B) P_M + [1 - G(\mu_B)] \frac{B_0}{\mu_B}$.

Let $\underline{P}_M$ be the unique solution to $P_M = \frac{\alpha Z}{\mu_B(P_M)}$. If $\frac{Z}{\mu^{\max}} \geq \underline{P}_M$, then $P_M = \underline{P}_M$ and $\lambda_M = 0$ is an intersection of the system. If $\frac{Z}{\mu^{\max}} < \underline{P}_M$, then note that at $\lambda_M = \lambda_E$, the inverse supply is $P_M = Z$, while the inverse demand is $P_M = \frac{\lambda_E Z + (1-\lambda_E)\alpha Z}{\mu_B} < Z$. Hence, an interior intersection exists. Since λ_M^S and λ_M^D are continuous, the set of intersections is compact, so a maximum volume of trade equilibrium exists and is unique.

Next, I show that in the maximum volume of trade equilibrium, $\frac{\partial \lambda_M^D}{\partial P_M} \geq \frac{\partial \lambda_M^S}{\partial P_M}$. Suppose, to the contrary, that $\frac{\partial \lambda_M^D}{\partial P_M} < \frac{\partial \lambda_M^S}{\partial P_M}$. It is straightforward to see that $\left(\lambda_M^D\right)^{-1}(\lambda_E) < \left(\lambda_M^S\right)^{-1}(\lambda_E)$. This means that if $\frac{\partial \lambda_M^D}{\partial P_M} < \frac{\partial \lambda_M^S}{\partial P_M}$ at the equilibrium price P_M^*, there exists $\widetilde{P}_M > P_M^*$ such that $\lambda_M^D = \lambda_M^S$. But this would imply that P_M^* is not part of the maximum volume of trade equilibrium, a contradiction.

Next, note that $\frac{\partial \mu_B}{\partial P_M} = \frac{[1-G(\mu_B)]H_B + \left[1 - G\left(\frac{Z}{P_M}\right)\right] H_G + g\left(\frac{Z}{P_M}\right)\frac{Z}{P_M} H_G}{g(\mu_B)\left[W + H_B P_M + \frac{B_0}{\mu_B} + \frac{1-G(\mu_B)}{g(\mu_B)\mu_B}\frac{B_0}{\mu_B}\right]} > 0$. Thus, since λ_M^D is decreasing in α and $\frac{\partial \lambda_M^D}{\partial P_M} \geq \frac{\partial \lambda_M^S}{\partial P_M}$, P_M, λ_M and μ_B are increasing in α. Similarly, since μ_B is decreasing in λ_E as a function of P_M, λ_M^D is decreasing in λ_E while λ_M^S is increasing. Since $\frac{\partial \lambda_M^D}{\partial P_M} \geq \frac{\partial \lambda_M^S}{\partial P_M}$, P_M and λ_M are increasing in λ_E. To see that μ_B is also increasing in λ_E, rewrite the market clearing condition as $G(\mu_B) W_1 = \left[\frac{1}{1-\lambda_M} - G(\mu_B)\right] H_B P_M + [1 - G(\mu_B)] \frac{B_0}{\mu_B}$. The LHS is increasing in μ_B, while the RHS is decreasing in μ_B. Moreover, the RHS is increasing in λ_M and P_M, hence μ_B is increasing in λ_E.

Finally, I need to show that if λ_E is sufficiently high, $P_M(X)$ is discontinuous at α^*. I do this by showing that $P_M = \frac{Z}{\mu^{\max}}$ cannot be part of an equilibrium. Suppose that $P_M = \frac{Z}{\mu^{\max}}$ were part of an equilibrium. This can happen only when $\alpha = \frac{\mu_B}{\mu^{\max}}$. Recall that a necessary condition for $P_M = \frac{Z}{\mu^{\max}}$ to be the maximum volume trade equilibrium is that $\frac{\partial \lambda_M^S}{\partial P_M}\big|_{P_M = \frac{Z}{\mu^{\max}}} \leq \frac{\partial \lambda_M^D}{\partial P_M}\big|_{P_M = \frac{Z}{\mu^{\max}}}$. We have $\frac{\partial \lambda_M^S}{\partial P_M}\big|_{P_M = \frac{Z}{\mu^{\max}}} = \frac{\lambda_E}{1-\lambda_E}\frac{g(\mu^{\max})(\mu^{\max})^2}{Z}$, while $\frac{\partial \lambda_M^D}{\partial P_M}\big|_{P_M = \frac{Z}{\mu^{\max}}} = \frac{\mu_B + \frac{Z}{\mu^{\max}}\frac{\partial \mu_B}{\partial P_M}\big|_{P_M = \frac{Z}{\mu^{\max}}}}{\left(1 - \frac{\mu_B}{\mu^{\max}}\right) Z}$, where $\frac{\partial \mu_B}{\partial P_M}\big|_{P_M = \frac{Z}{\mu^{\max}}} = \frac{[1-G(\mu_B)]H_B + g(\mu^{\max})\mu^{\max} H_G}{g(\mu_B)\left[W + H_B \frac{Z}{\mu^{\max}} + \frac{B_0}{\mu_B} + \frac{1-G(\mu_B)}{g(\mu_B)\mu_B}\frac{B_0}{\mu_B}\right]}$. Hence $\frac{\partial \lambda_M^S}{\partial P_M}\big|_{P_M = \frac{Z}{\mu^{\max}}} < \frac{\partial \lambda_M^D}{\partial P_M}\big|_{P_M = \frac{Z}{\mu^{\max}}}$ if and only if

$$\frac{\lambda_E}{1-\lambda_E}\frac{g(\mu^{\max})(\mu^{\max})^2}{Z} < \frac{\mu_B + \frac{Z}{\mu^{\max}} \frac{[1-G(\mu_B)]H_B + g(\mu^{\max})\mu^{\max} H_G}{g(\mu_B)\left[W + H_B \frac{Z}{\mu^{\max}} + \frac{B_0}{\mu_B} + \frac{1-G(\mu_B)}{g(\mu_B)\mu_B}\frac{B_0}{\mu_B}\right]}}{\left(1 - \frac{\mu_B}{\mu^{\max}}\right) Z}$$

But as $\lambda_E \to 1$, the LHS goes to infinity as long as $g(\mu^{\max}) > 0$, and the RHS is bounded (since $\mu_B < \mu^{\max}$). Thus, for λ_E sufficiently high, there does not exist a maximum volume of trade equilibrium with $P_M = \frac{Z}{\mu^{\max}}$.

Finally, if the cdf G is weakly convex and log-concave, then the maximum volume of trade equilibrium is continuous in α except at $\alpha = \alpha^*$. To see this, note that $\frac{\partial \lambda_M^S}{\partial P_M} = -\left[2\left(\frac{g\left(\frac{Z}{P_M}\right)\frac{Z}{P_M^2} H_G}{\left[1 - G\left(\frac{Z}{P_M}\right)\right] H_G + H_B} + \frac{1}{P_M}\right) + \frac{g'\left(\frac{Z}{P_M}\right)}{g\left(\frac{Z}{P_M}\right)}\frac{Z}{P_M^2}\right]\frac{\partial \lambda_M^S}{\partial P_M}$, which is negative under Assumption 2. Moreover, $\frac{\partial^2 \lambda_M^D}{\partial P_M^2} = \frac{\frac{\partial^2 \mu_B}{\partial P_M^2} P_M + 2\frac{\partial \mu_B}{\partial P_M}}{(1-\alpha)Z}$, where

$$\frac{\partial^2 \mu_B}{\partial P_M^2} P_M + 2\frac{\partial \mu_B}{\partial P_M} = \left[\frac{\partial \frac{G(\mu_B)}{g(\mu_B)}}{\partial P_M}\left[1 - (1-\lambda_M) G(\mu_B)\right] + \left(1 - G(\mu_B)(1-\lambda_M)\right)\right]\frac{\partial \mu_B}{\partial P_M}$$

which is positive under Assumption 2. Hence, the demand and supply can intersect at most twice when $\lambda_M > 0$, and the maximum volume of trade equilibrium is continuous in α.

□

Proof of Lemma 3. Since the cdf of α, F, is continuous, the shadow values of good and bad trees are continuous in λ_E even if market prices are discontinuous in the state of the economy. Because H_G is a continuous function of the shadow values, the function T is continuous in λ_E. Moreover, since prices are increasing in λ_E, Lemma 2 implies that H_G is decreasing in λ_E, so T is decreasing in λ_E. Therefore, by Brouwer fixed-point theorem, a fixed-point of T exists and is unique. That $\lambda_E \in (0,1)$ follows from Assumption 1 and that $C'(W_0) > 1$. □

Proof of Proposition 4. We have

$$\gamma_G = \int_{\underline{\alpha}}^{\overline{\alpha}} \left[G\left(\frac{Z}{P_M(X)}\right) Z + \int_{\frac{Z}{P_M(X)}}^{\mu^{\max}} \mu P_M(X) dG(\mu) \right] dF(\alpha|\theta)$$

$$\gamma_B = \int_{\underline{\alpha}}^{\overline{\alpha}} \left[G\left(\mu_B(P_M(X),X)\right) \mu_B(P_M(X),X) P_M(X) + \int_{\mu_B(P_M(X),X)}^{\mu^{\max}} \mu P_M(X) dG(\mu) \right] dF(\alpha|\theta)$$

where $\mu_B(P_M(X),X)$ is implicitly defined as the solution to

$$G(\mu_B)W_1 = [1-G(\mu_B)]H_B P_M(X) + \left[1 - G\left(\frac{Z}{P_M(X)}\right)\right] H_G P_M(X) + [1-G(\mu_B)]\frac{B_0}{\mu_B}$$

and $P_M(X)$ is then the solution to

$$P_M = \frac{\lambda_M(P_M,X)Z + (1-\lambda_M(P_M,X)\alpha Z}{\mu_B(P_M,X)}, \quad \lambda_M(P_M,X) = \frac{\left[1-G\left(\frac{Z}{P_M}\right)\right]\lambda_E}{\left[1-G\left(\frac{Z}{P_M}\right)\right]\lambda_E + (1-\lambda_E)}.$$

Since P_M is increasing in α for any value of λ_E, the increase in $F(\cdot|\theta)$ to $F(\cdot|\theta')$ is mathematically equivalent to an increase in prices in each state α by $\phi(\alpha;\lambda_E) \geq 0$. By Lemma 2, an increase in prices reduces $H_G(\lambda_E)$, so that the function $T(\lambda_E)$ decreases for all λ_E. Hence, the fixed point $\lambda_E^* = T(\lambda_E^*)$ decreases.

Because bad trees are cheaper to produce than good trees, the total production of trees increases. Moreover, because λ_E decreases, equilibrium prices decrease in all states, so the threshold α^* increases. Finally, market fragility is ambiguous since the change in F reduces it but the endogenous change in λ_E increases it. □

Proof of Lemma 4. First, note that an increase in B_0 reduces the net demand for trees. For states with $\alpha < \alpha^*$, the result is an increase in $\mu_B = r_M$, a drop in the price of trees P_M, but an increase in total liquidity, since total demand for assets is higher. For states with $\alpha \geq \alpha^*$, the response depends on on whether α is in the neighborhood of α^*. Consider the state $\alpha = \alpha^*$. The increase in B_0 pushes the demand down, so that an equilibrium in the market for trees with $\lambda_M > 0$ ceases to exist. In that case, P_M discontinuously drops. If the change in B_0 is small, then total liquidity decreases (since private liquidity drops discretely). Finally, if α is sufficiently higher than α^*, the drop in demand does not trigger a market collapse, so $\mu_B = r_M$ increases and total liquidity increases. Since in all cases P_M decreases, λ_M also decreases. □

Proof of Proposition 5. From Lemma 4 we know that, given $\{\lambda_E, H\}$, P_M and $\mu_B P_M = \lambda_M Z + (1-\lambda_M)\alpha Z$ decrease in all states. Thus, $H_G(\lambda_E)$ increases and the function T increases. Therefore, equilibrium λ_E increases. The effect on market fragility is $\frac{dMF}{dB_0} \propto \underbrace{\frac{\partial \mathrm{MF}}{\partial B_0}}_{>0} + \underbrace{\frac{\partial \mathrm{MF}}{\partial \lambda_E}}_{<0} \underbrace{\frac{\partial \lambda_E}{\partial B_0}}_{>0}$. Note that we have $\frac{\partial \lambda_E}{\partial B_0} = \frac{\lambda_E C'(H_G^*) + (1-\lambda_E)}{H^*} \frac{\partial H_G}{\partial B_0}$ and $\frac{\partial H_G}{\partial B_0} = \frac{\frac{\partial \gamma_G}{\partial B_0}\frac{1}{\gamma_G} - \frac{\partial \gamma_B}{\partial B_0}\frac{1}{\gamma_B}}{\eta_G^{-1} - \frac{\partial \gamma_G}{\partial H_G}\frac{1}{\gamma_G} + \frac{\partial \gamma_B}{\partial H_G}\frac{1}{\gamma_B}} > 0 \implies \frac{\partial \lambda_E}{\partial B_0} > 0$, where starred variables denote equilibrium variables and $\eta_G \equiv \frac{C'(H_G^*)}{C''(H_G^*)}$. Then $\frac{\partial^2 \lambda_E}{\partial B_0 \partial \eta_G} = \frac{\lambda_E C'(H_G^*) + (1-\lambda_E)}{H^*} \frac{\partial^2 H_G}{\partial B_0 \partial \eta_G}$ and $\frac{\partial^2 H_G}{\partial B_0 \partial \eta_G} = \frac{\frac{\partial \gamma_G}{\partial B_0}\frac{1}{\gamma_G} - \frac{\partial \gamma_B}{\partial B_0}\frac{1}{\gamma_B}}{\left[\eta_G^{-1} - \frac{\partial \gamma_G}{\partial H_G}\frac{1}{\gamma_G} + \frac{\partial \gamma_B}{\partial H_G}\frac{1}{\gamma_B}\right]^2} \eta_G^{-2} < 0 \implies \frac{\partial^2 \lambda_E}{\partial B_0 \partial \eta_G} > 0$. Then $\frac{d^2 \mathrm{MF}}{dB_0 d\eta_G} = \frac{\partial \mathrm{MF}}{\partial \lambda_E} \frac{\partial^2 \lambda_E}{\partial B_0 \partial \eta_G} < 0$. □

Proof of Proposition 6. Consider the Ramsey problem. Denote the solution by $\{\hat{B}(\alpha), \hat{\omega}(\alpha)\}$. Suppose the solution satisfies $0 > \mathbb{T}_{\mathbf{2}}(\alpha) > -W_2$ for all α, and $\boldsymbol{U}_{\mathbf{1}}(\alpha)$ and $\mathbb{T}_{\mathbf{2}}(\alpha)$ are differentiable at $\{\hat{B}(\alpha), \hat{\omega}(\alpha)\}$. Consider the alternative policy

$$\tilde{B}(\alpha,\varepsilon) = \hat{B}(\alpha) + \epsilon \eta_1(\alpha) \quad \text{and} \quad \tilde{\omega}(\alpha,\varepsilon) = \hat{\omega}(\alpha) + \epsilon \eta_2(\alpha)$$

for some arbitrary functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ and ε sufficiently small so that the constraints on $\mathbb{T}_2(\cdot)$ are satisfied with strict inequality. Moreover, note that the determination of H_G implies

$$H_G(\varepsilon) = \boldsymbol{H}_{\boldsymbol{G}}(\{\tilde{B}(\alpha,\varepsilon), \tilde{\omega}(\alpha,\varepsilon)\}_{\forall \alpha})$$

Define

$$\boldsymbol{W}(\varepsilon) \equiv \int_0^1 \left[\boldsymbol{U}_{\mathbf{1}}(\tilde{B}(\alpha,\varepsilon), \tilde{\omega}(\alpha,\varepsilon), H_G(\varepsilon)) + Z H_G(\varepsilon) + \alpha Z(W - C(H_G(\varepsilon))) + \frac{\chi}{1+\chi} \mathbb{T}_{\mathbf{2}}(\tilde{B}(\alpha,\varepsilon), \tilde{\omega}(\alpha,\varepsilon), H_G(\varepsilon)) \right] dF(\alpha)$$

Naturally, $W(\cdot)$ is maximized at $\varepsilon = 0$. Note that for any arbitrary scalar λ, we have

$$\lambda\left[\boldsymbol{H_G}(\{\tilde{B}(\alpha,\varepsilon),\tilde{\omega}(\alpha,\varepsilon)\}_{\forall\alpha}) - H_G(\varepsilon)\right] = 0$$

Adding this last expression to $W(\varepsilon)$, we get

$$\boldsymbol{W}(\varepsilon) = \int_0^1 \Big[\boldsymbol{U_1}(\tilde{B}(\alpha,\varepsilon),\tilde{\omega}(\alpha,\varepsilon),H_G(\varepsilon)) + ZH_G(\varepsilon) + \alpha Z(W - C(H_G(\varepsilon)))+ \\ \frac{\chi}{1+\chi}\mathbb{T}_{\boldsymbol{2}}(\tilde{B}(\alpha,\varepsilon),\tilde{\omega}(\alpha,\varepsilon),H_G(\varepsilon))\Big]\, dF(\alpha) + \lambda\left[\boldsymbol{H_G}(\{\tilde{B}(\alpha,\varepsilon),\tilde{\omega}(\alpha,\varepsilon)\}_{\forall\alpha}) - H_G(\varepsilon)\right]$$

Differentiating with respect to ε, we get

$$\boldsymbol{W}'(\varepsilon) = \int_0^1 \Bigg[\frac{\partial\boldsymbol{U_1}(\alpha,\varepsilon)}{\partial B(\alpha)}\eta_1(\alpha) + \frac{\partial\boldsymbol{U_1}(\alpha,\varepsilon)}{\partial\omega(\alpha)}\eta_2(\alpha) + \frac{\partial\boldsymbol{U_1}(\alpha,\varepsilon)}{\partial H_G}H_G'(\varepsilon) + (1-\alpha C'(H_G(\varepsilon)))ZH_G'(\varepsilon)+ \\ \frac{\chi}{1+\chi}\left[\frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha,\varepsilon)}{\partial B(\alpha)}\eta_1(\alpha) + \frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha,\varepsilon)}{\partial\omega(\alpha)}\eta_2(\alpha) + \frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha,\varepsilon)}{\partial H_G}H_G'(\varepsilon)\right]\Bigg]\, dF(\alpha)+ \\ \lambda\left[\int_0^1\left[\frac{\partial\boldsymbol{H_G}(\alpha,\varepsilon)}{\partial B(\alpha)}\eta_1(\alpha) + \frac{\partial\boldsymbol{H_G}(\alpha,\varepsilon)}{\partial\omega(\alpha)}\eta_2(\alpha)\right]d\alpha - H_G'(\varepsilon)\right]$$

or

$$\boldsymbol{W}'(\varepsilon) = \int_0^1\left[\left[\frac{\partial\boldsymbol{U_1}(\alpha,\varepsilon)}{\partial B(\alpha)} + \frac{\chi}{1+\chi}\frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha,\varepsilon)}{\partial B(\alpha)}\right]f(\alpha) + \lambda\frac{\partial\boldsymbol{H_G}(\alpha,\varepsilon)}{\partial B(\alpha)}\right]\eta_1(\alpha)d\alpha+ \\ \int_0^1\left[\left[\frac{\partial\boldsymbol{U_1}(\alpha,\varepsilon)}{\partial\omega(\alpha)} + \frac{\chi}{1+\chi}\frac{\partial\mathbb{T}_{\boldsymbol{2}}|(\alpha,\varepsilon)}{\partial\omega(\alpha)}\right]f(\alpha) + \lambda\frac{\partial\boldsymbol{H_G}(\alpha,\varepsilon)}{\partial\omega(\alpha)}\right]\eta_2(\alpha)d\alpha+ \\ \left[\int_0^1\left[\frac{\partial\boldsymbol{U_1}(\alpha,\varepsilon)}{\partial H_G} + (1-\alpha C'(H_G(\varepsilon)))Z + \frac{\chi}{1+\chi}\frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha,\varepsilon)}{\partial H_G} - \right]dF(\alpha) - \lambda\right]H_G'(\varepsilon)$$

Optimality requires that

$$\boldsymbol{W}'(0) = 0.$$

Since the choice of λ was arbitrary, I specialize to the following expression

$$\lambda = E\left[\frac{\partial\boldsymbol{U_1}(\alpha)}{\partial H_G} + (1-\alpha C'(H_G))Z - \frac{\chi}{1+\chi}\frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha)}{\partial H_G}\right]$$

Moreover, since the choice of the functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ was also arbitrary, we obtain $W'(0) = 0$ if and only if

$$\left[\frac{\partial\boldsymbol{U_1}(\alpha)}{\partial B(\alpha)} + \frac{\chi}{1+\chi}\frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha)}{\partial B(\alpha)}\right]f(\alpha) + E\left[\frac{\partial\boldsymbol{U_1}(\alpha)}{\partial H_G} + (1-\alpha C'(H_G))Z + \frac{\chi}{1+\chi}\frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha)}{\partial H_G}\right]\frac{\partial\boldsymbol{H_G}(\alpha)}{\partial B(\alpha)} = 0$$

and

$$\left[\frac{\partial\boldsymbol{U_1}(\alpha)}{\partial\omega(\alpha)} + \frac{\chi}{1+\chi}\frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha)}{\partial\omega(\alpha)}\right]f(\alpha) + E\left[\frac{\partial\boldsymbol{U_1}(\alpha)}{\partial H_G} + (1-\alpha C'(H_G))Z + \frac{\chi}{1+\chi}\frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha)}{\partial H_G}\right]\frac{\partial\boldsymbol{H_G}(\alpha)}{\partial\omega(\alpha)} = 0$$

For $\boldsymbol{\lambda_E} \in (0,1)$, the total derivative of $\boldsymbol{H_G}$ is given by

$$\eta_G\left(\boldsymbol{H_G}\right)d\boldsymbol{H_G} = \frac{d\boldsymbol{\gamma_G}}{\boldsymbol{\gamma_G}} - \frac{d\boldsymbol{\gamma_B}}{\boldsymbol{\gamma_B}}$$

where $\eta_G\left(\boldsymbol{H_G}\right) = \frac{C''(\boldsymbol{H_G})}{C'(\boldsymbol{H_G})}$. Let

$$\overline{\boldsymbol{\gamma}}_{\boldsymbol{G}}(\alpha) \equiv \int_1^{\frac{Z}{(1+\omega(\alpha))\boldsymbol{P_B}(\alpha)}} ZdG(\mu) + \int_{\frac{Z}{(1+\omega(\alpha))\boldsymbol{P_B}(\alpha)}}^{\mu^{\max}} \mu\left(1+\omega(\alpha)\right)\boldsymbol{P_B}(\alpha)\,dG(\mu)$$

$$\overline{\boldsymbol{\gamma}}_{\boldsymbol{B}}(\alpha) \equiv \int_1^{\boldsymbol{\mu_B}(\alpha)} \boldsymbol{\mu_B}(\alpha)\left(1+\omega(\alpha)\right)\boldsymbol{P_B}(\alpha)\,dG(\mu) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu\left(1+\omega(\alpha)\right)\boldsymbol{P_B}(\alpha)\,dG(\mu).$$

Then

$$d\boldsymbol{\gamma_G} = \int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial B(\alpha)} dB(\alpha)\, dF(\alpha) + \int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial \omega(\alpha)} d\omega(\alpha)\, dF(\alpha) + \int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G} d\boldsymbol{H_G} dF(\alpha)$$

$$d\boldsymbol{\gamma_B} = \int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial B(\alpha)} dB(\alpha)\, dF(\alpha) + \int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial \omega(\alpha)} d\omega(\alpha)\, dF(\alpha) + \int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G} d\boldsymbol{H_G} dF(\alpha)$$

Putting everything together, we get

$$\frac{\partial \boldsymbol{H_G}}{\partial B(\alpha)} = \frac{\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial B(\alpha)} \frac{1}{\gamma_G} - \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial B(\alpha)} \frac{1}{\gamma_B}}{\eta_G(\boldsymbol{H_G}) - \left[\frac{\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G} dF(\alpha)}{\gamma_G} - \frac{\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G} dF(\alpha)}{\gamma_B} \right]} f(\alpha)$$

$$\frac{\partial \boldsymbol{H_G}}{\partial \omega(\alpha)} = \frac{\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial \omega(\alpha)} \frac{1}{\gamma_G} - \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial \omega(\alpha)} \frac{1}{\gamma_B}}{\eta_G(\boldsymbol{H_G}) - \left[\frac{\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G} dF(\alpha)}{\gamma_G} - \frac{\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G} dF(\alpha)}{\gamma_B} \right]} f(\alpha)$$

where

$$\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial B(\alpha)} = \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu(1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial B(\alpha)} dG(\mu)$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial B(\alpha)} = \int_1^{\boldsymbol{\mu_B}(\alpha)} (1+\omega(\alpha)) \left[\frac{\partial \boldsymbol{\mu_B}(\alpha)}{\partial B(\alpha)} \boldsymbol{P_B}(\alpha) + \boldsymbol{\mu_B}(\alpha) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial B(\alpha)} \right] dG(\mu) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu(1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial B(\alpha)} dG(\mu)$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial \omega(\alpha)} = \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu \left[\boldsymbol{P_B}(\alpha) + (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial \omega(\alpha)} \right] dG(\mu)\, f(\alpha)$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial \omega(\alpha)} = \left[\int_1^{\boldsymbol{\mu_B}(\alpha)} \left[\frac{\partial \boldsymbol{\mu_B}(\alpha)}{\partial \omega(\alpha)} (1+\omega(\alpha)) \boldsymbol{P_B}(\alpha) + \boldsymbol{P_B}(\alpha) + (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial \omega(\alpha)} \right] dG(\mu) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu \left[\boldsymbol{P_B}(\alpha) + (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial \omega(\alpha)} \right] dG(\mu) \right] f(\alpha)$$

$$\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G} dF(\alpha) = \int_0^1 \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu(1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial H_G} dG(\mu)\, dF(\alpha)$$

$$\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G} dF(\alpha) = \int_0^1 \left[\int_1^{\boldsymbol{\mu_B}(\alpha)} (1+\omega(\alpha)) \left[\frac{\partial \boldsymbol{\mu_B}(\alpha)}{\partial H_G} \boldsymbol{P_B}(\alpha) + \boldsymbol{\mu_B}(\alpha) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial H_G} \right] dG(\mu) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu(1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial H_G} dG(\mu) \right] dF(\alpha)$$

Consider $\frac{\partial \boldsymbol{H}_G}{\partial B(\alpha)}$. After some algebra, we get

$$\frac{\partial \boldsymbol{\mu_B}(\alpha)}{\partial B(\alpha)} > 0, \quad \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial B(\alpha)} < 0, \quad \frac{\partial \boldsymbol{\lambda_M}(\alpha)}{\partial B(\alpha)} < 0$$

which implies that $\frac{\partial \boldsymbol{\mu_B}(\alpha)}{\partial B(\alpha)} \boldsymbol{P_B}(\alpha) + \boldsymbol{\mu_B}(\alpha) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial B(\alpha)} < 0$, and

$$\frac{\partial \boldsymbol{P_B}(\alpha)}{\partial H_G} > 0, \quad \frac{\partial \boldsymbol{\lambda_M}(\alpha)}{\partial H_G} > 0$$

which implies that $\frac{\partial \boldsymbol{\mu_B}(\alpha)}{\partial H_G} \boldsymbol{P_B}(\alpha) + \boldsymbol{\mu_B}(\alpha) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial H_G} > 0$.Then

$$\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial B(\alpha)} = \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu(1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial B(\alpha)} dG(\mu) \leq 0$$

and

$$\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial B(\alpha)} = \int_{1}^{\boldsymbol{\mu_B}(\alpha)} (1+\omega(\alpha)) \left[\frac{\partial \boldsymbol{\mu_B}(\alpha)}{\partial B(\alpha)} \boldsymbol{P_B}(\alpha) + \boldsymbol{\mu_B}(\alpha) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial B(\alpha)} \right] dG(\mu) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial B(\alpha)} dG(\mu) < \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial B(\alpha)} dG(\mu) \leq 0$$

hence, $\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial B(\alpha)} \frac{1}{\boldsymbol{\gamma_G}} - \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial B(\alpha)} \frac{1}{\boldsymbol{\gamma_B}} > 0$. Moreover,

$$\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G} dF(\alpha) = \int_0^1 \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial H_G} dG(\mu) dF(\alpha) \geq 0$$

$$\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G} dF(\alpha) = \int_0^1 \left[\int_{1}^{\boldsymbol{\mu_B}(\alpha)} (1+\omega(\alpha)) \left[\frac{\partial \boldsymbol{\mu_B}(\alpha)}{\partial H_G} \boldsymbol{P_B}(\alpha) + \boldsymbol{\mu_B}(\alpha) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial H_G} \right] dG(\mu) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial H_G} dG(\mu) \right] dF(\alpha) > \int_0^1 \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial H_G} dG(\mu) dF(\alpha) \geq 0$$

hence $\frac{\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G} dF(\alpha)}{\boldsymbol{\gamma_G}} - \frac{\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G} dF(\alpha)}{\boldsymbol{\gamma_B}} < 0$. Therefore

$$\frac{\partial \boldsymbol{H_G}}{\partial B(\alpha)} = \frac{\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial B(\alpha)} \frac{1}{\boldsymbol{\gamma_G}} - \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial B(\alpha)} \frac{1}{\boldsymbol{\gamma_B}}}{\eta_G(\boldsymbol{H_G}) - \left[\frac{\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G} dF(\alpha)}{\boldsymbol{\gamma_G}} - \frac{\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G} dF(\alpha)}{\boldsymbol{\gamma_B}} \right]} f(\alpha) > 0$$

Consider now $\frac{\partial \boldsymbol{H_G}}{\partial \omega(\alpha)}$. After some algebra, we get

$$\frac{\partial \boldsymbol{\mu_B}(\alpha)}{\partial \omega(\alpha)} > 0, \quad (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial \omega(\alpha)} + \boldsymbol{P_B}(\alpha) > 0.$$

Then

$$\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial \omega(\alpha)} = \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu \left[\boldsymbol{P_B}(\alpha) + (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial \omega(\alpha)} \right] dG(\mu) f(\alpha) \geq 0$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial \omega(\alpha)} = \left[\int_{1}^{\boldsymbol{\mu_B}(\alpha)} \left[\frac{\partial \boldsymbol{\mu_B}(\alpha)}{\partial \omega(\alpha)} (1+\omega(\alpha)) \boldsymbol{P_B}(\alpha) + \boldsymbol{P_B}(\alpha) + (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial \omega(\alpha)} \right] dG(\mu) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu \left[\boldsymbol{P_B}(\alpha) + (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial \omega(\alpha)} \right] dG(\mu) \right] f(\alpha) > \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu \left[\boldsymbol{P_B}(\alpha) + (1+\omega(\alpha)) \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial \omega(\alpha)} \right] dG(\mu) f(\alpha) \geq 0$$

hence $\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial \omega(\alpha)} \frac{1}{\boldsymbol{\gamma_G}} - \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial \omega(\alpha)} \frac{1}{\boldsymbol{\gamma_B}} < 0$, and therefore

$$\frac{\partial \boldsymbol{H_G}}{\partial \omega(\alpha)} = \frac{\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial \omega(\alpha)} \frac{1}{\boldsymbol{\gamma_G}} - \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial \omega(\alpha)} \frac{1}{\boldsymbol{\gamma_B}}}{\eta_G(\boldsymbol{H_G}) - \left[\frac{\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G} dF(\alpha)}{\boldsymbol{\gamma_G}} - \frac{\int_0^1 \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G} dF(\alpha)}{\boldsymbol{\gamma_B}} \right]} f(\alpha) < 0.$$

Finally, start at $B(\alpha) = \omega(\alpha) = 0$. Then

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial B(\alpha)} = \int_{\mu_B(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) \frac{1}{\boldsymbol{\mu_B}(\alpha)} dG(\mu) + \left[\int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) \boldsymbol{H_B} dG(\mu) + \int_{\mu_S(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) H_G dG(\mu) + (\boldsymbol{\mu_S}(\alpha) - \boldsymbol{\mu}_B(\alpha)) g(\boldsymbol{\mu_S}(\alpha)) \boldsymbol{\mu_S}(\alpha) H_G\right] \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial B(\alpha)}$$

$$\frac{\partial \mathbb{T}_2(\alpha)}{\partial B(\alpha)} = (1+\chi)$$

and

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial \omega(\alpha)} = \left[\int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) \boldsymbol{H_B P_B}(\alpha) dG(\mu) + \int_{\mu_S(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) H_G \boldsymbol{P_B}(\alpha) dG(\mu) + (\boldsymbol{\mu_S}(\alpha) - \boldsymbol{\mu}_B(\alpha)) g(\boldsymbol{\mu_S}(\alpha)) \boldsymbol{\mu_S}(\alpha) H_G \boldsymbol{P_B}(\alpha)\right] + \left[\int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) \boldsymbol{H_B} dG(\mu) + \int_{\mu_S(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) H_G dG(\mu) + (\boldsymbol{\mu_S}(\alpha) - \boldsymbol{\mu}_B(\alpha)) g(\boldsymbol{\mu_S}(\alpha)) \boldsymbol{\mu_S}(\alpha) H_G\right] \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial \omega(\alpha)}$$

$$\frac{\partial \mathbb{T}_2(\alpha)}{\partial \omega(\alpha)} = (1+\chi)\left[\boldsymbol{\mu_B}(\alpha) \boldsymbol{P_B}(\alpha) \boldsymbol{S}(\alpha)\right]$$

Then $\frac{\partial \mathbb{T}_2(\alpha)}{\partial B(\alpha)} = \frac{\partial \mathbb{T}_2(\alpha)}{\partial \omega(\alpha)} d\omega(\alpha)$ if and only if

$$d\omega(\alpha) = \frac{1}{\boldsymbol{\mu_B}(\alpha) \boldsymbol{P_B}(\alpha) \boldsymbol{S}(\alpha)}$$

Since $\frac{\partial \boldsymbol{P_B}(\alpha)}{\partial \omega(\alpha)} > 0 > \frac{\partial \boldsymbol{P_B}(\alpha)}{\partial B(\alpha)}$, to show that $\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial B(\alpha)} / \frac{\partial \mathbb{T}_2(\alpha)}{\partial B(\alpha)} \geq \frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial \omega(\alpha)} / \frac{\partial \mathbb{T}_2(\alpha)}{\partial B(\alpha)}$ it is sufficient to show that

$$\left[\int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) \boldsymbol{H_B P_B}(\alpha) dG(\mu) + \int_{\mu_S(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) H_G \boldsymbol{P_B}(\alpha) dG(\mu) + (\boldsymbol{\mu_S}(\alpha) - \boldsymbol{\mu}_B(\alpha)) g(\boldsymbol{\mu_S}(\alpha)) \boldsymbol{\mu_S}(\alpha) H_G \boldsymbol{P_B}(\alpha)\right] d\omega(\alpha) > \int_{\mu_B(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) \frac{1}{\boldsymbol{\mu_B}(\alpha)} dG(\mu)$$

After some algebra, we get that the previous condition holds if and only if

$$[1 - G(\boldsymbol{\mu_S}(\alpha))] \left[\frac{\int_{\mu_S(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) dG(\mu)}{1 - G(\boldsymbol{\mu_S}(\alpha))} - [1 - G(\boldsymbol{\mu_B}(\alpha))] \frac{\int_{\mu_B(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu}_B(\alpha)) dG(\mu)}{1 - G(\boldsymbol{\mu_B}(\alpha))}\right] + (\boldsymbol{\mu_S}(\alpha) - \boldsymbol{\mu}_B(\alpha)) g(\boldsymbol{\mu_S}(\alpha)) \boldsymbol{\mu_S}(\alpha) > 0$$

which is correct. Similar calculations show that these result also hold when $\tilde{\boldsymbol{\mu}}_{\boldsymbol{B}}(\alpha) = \frac{\alpha Z}{(1+\omega(\alpha))\boldsymbol{P_B}(\alpha)}$.

□

Proof of Proposition 7. It will be convenient to characterize the solution in terms of $P_S(\alpha)$, where $P_S(\alpha) = (1+\omega(\alpha)) P_B(\alpha)$, and $\overline{T}_1(\alpha)$, where $\overline{T}_1(\alpha) = \frac{B(\alpha)}{\boldsymbol{\mu_B}(\alpha)}$. Note that

$$\frac{d\boldsymbol{H_G}}{dP_S(\alpha)} = \frac{\frac{\partial\overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial P_S(\alpha)}\frac{1}{C'(\boldsymbol{H_G})} - \frac{\partial\overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial P_S(\alpha)}}{\eta_G(\boldsymbol{H_G})\boldsymbol{\gamma_B} + E\left[\frac{\partial\overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G}\right] - C'(\boldsymbol{H_G})E\left[\frac{\partial\overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G}\right]}f(\alpha),$$

$$\frac{d\boldsymbol{H_G}}{d\overline{T}_1(\alpha)} = \frac{\frac{\partial\overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial\overline{T}_1(\alpha)}\frac{1}{C'(\boldsymbol{H_G})} - \frac{\partial\overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial\overline{T}_1(\alpha)}}{\eta_G(\boldsymbol{H_G})\boldsymbol{\gamma_B} + E\left[\frac{\partial\overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G}\right] - C'(\boldsymbol{H_G})E\left[\frac{\partial\overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G}\right]}f(\alpha)$$

where $\eta_G(\boldsymbol{H_G}) \equiv \frac{C''(\boldsymbol{H_G})}{C'(\boldsymbol{H_G})}$,

$$\overline{\gamma}_{\boldsymbol{G}}(\alpha) \equiv \int_0^{\boldsymbol{\mu_S}(\alpha)} ZdG(\mu) + \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu P_S(\alpha)\,dG(\mu)$$

and

$$\overline{\gamma}_{\boldsymbol{B}}(\alpha) \equiv \int_0^{\tilde{\boldsymbol{\mu}}_{\boldsymbol{B}}(\alpha)} \tilde{\boldsymbol{\mu}}_{\boldsymbol{B}}(\alpha)\,P_S(\alpha)\,dG(\mu) + \int_{\tilde{\boldsymbol{\mu}}_{\boldsymbol{B}}(\alpha)}^{\mu^{\max}} \mu P_S(\alpha)\,dG(\mu).$$

Let

$$\tilde{\Omega} \equiv \frac{E\left[\frac{\partial\boldsymbol{\mathcal{U}_1}(\alpha)}{\partial H_G} + (1-\alpha C'(\boldsymbol{H_G}))Z + \frac{\chi}{1+\chi}\frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha)}{\partial H_G}\right]}{\eta_G(\boldsymbol{H_G})\boldsymbol{\gamma_B} + E\left[\frac{\partial\overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G}\right] - \frac{1}{C'(\boldsymbol{H_G})}E\left[\frac{\partial\overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G}\right]}.$$

Then, the necessary conditions for the optimality of $P_S(\alpha)$ and $\overline{T}_1(\alpha)$ are

$$\left[\frac{\partial\boldsymbol{\mathcal{U}_1}(\alpha)}{\partial P_S(\alpha)} - \chi\frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha)}{\partial P_S(\alpha)}\right] + \tilde{\Omega}\left[\frac{\partial\overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial P_S(\alpha)}\frac{1}{C'(\boldsymbol{H_G})} - \frac{\partial\overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial P_S(\alpha)}\right] = 0$$

and

$$\left[\frac{\partial\boldsymbol{\mathcal{U}_1}(\alpha)}{\partial\overline{T}_1(\alpha)} - \chi\frac{\partial\mathbb{T}_{\boldsymbol{2}}(\alpha)}{\partial\overline{T}_1(\alpha)}\right] + \tilde{\Omega}\left[\frac{\partial\overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial\overline{T}_1(\alpha)}\frac{1}{C'(\boldsymbol{H_G})} - \frac{\partial\overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial\overline{T}_1(\alpha)}\right] = 0$$

The characterization has three steps. First, I characterize the optimality conditions for $P_S(\alpha)$ and $\overline{T}_1(\alpha)$ for the different regions where the objective function is differentiable, given $\tilde{\Omega}$. Then, I characterize the the thresholds $\tilde{\alpha}$that divide crisis and normal states. Finally, I determine $\tilde{\Omega}$.

Since $\boldsymbol{\mathcal{U}_1}(\alpha)$ and $\mathbb{T}_{\boldsymbol{2}}(\alpha)$ are not everywhere differentiable, we need to characterize six seprate *market regimes*:

i. $P_S(\alpha) > \frac{Z}{\mu^{\max}}$ and $P_S(\alpha) > \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$

ii. $P_S(\alpha) > \frac{Z}{\mu^{\max}}$ and $P_S(\alpha) < \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$

iii. $P_S(\alpha) > \frac{Z}{\mu^{\max}}$ and $P_S = \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$

iv. $P_S(\alpha) < \frac{Z}{\mu^{\max}}$ and $P_S(\alpha) > \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$

v. $P_S(\alpha) < \frac{Z}{\mu^{\max}}$ and $P_S(\alpha) < \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$

vi. $P_S(\alpha) < \frac{Z}{\mu^{\max}}$ and $P_S(\alpha) = \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$

Regime i: $P_S(\alpha) > \frac{Z}{\mu^{\max}}$ and $P_S(\alpha) > \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$. For $\boldsymbol{\mathcal{U}_1}(\alpha)$, we have

$$\frac{\partial\boldsymbol{\mathcal{U}_1}(\alpha)}{\partial P_S(\alpha)} = \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\,\boldsymbol{H_B}dG(\mu) + \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\,H_G dG(\mu) + (\boldsymbol{\mu_S}(\alpha) - \boldsymbol{\mu_B}(\alpha))\,\boldsymbol{\mu_S}(\alpha)\,H_G g(\boldsymbol{\mu_S}(\alpha)) > 0$$

and

$$\frac{\partial\boldsymbol{\mathcal{U}_1}(\alpha)}{\partial\overline{T}_1(\alpha)} = \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\,dG(\mu) > 0$$

For transfers $\mathbb{T}_{\mathbf{2}}(\alpha)$, we have

$$\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial P_S(\alpha)} = (1+\chi)\left[\left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\right)[[1-G(\boldsymbol{\mu_S}(\alpha))] + g(\boldsymbol{\mu_S}(\alpha))\,\boldsymbol{\mu_S}(\alpha)]\,\boldsymbol{H_G} + \left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}[1-G(\boldsymbol{\mu_B}(\alpha))]\right)\boldsymbol{H_B} - \boldsymbol{\mu_S}(\alpha) g(\boldsymbol{\mu_S}(\alpha))\,\boldsymbol{\mu_S}(\alpha)\,\boldsymbol{H_G}\right]$$

and

$$\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial \overline{T}_1(\alpha)} = (1+\chi)\left[\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}[1-G(\boldsymbol{\mu_B}(\alpha))]\right] > 0$$

Note that in $\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial P_S(\alpha)}$, the first two terms are strictly positive, while the last term is negative. For $\lambda_E \to 1$ and $P_S(\alpha) \to \frac{Z}{\mu^{\max}}$, we get $\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial P_S(\alpha)} < 0$. As $P_S(\alpha) \to Z$, we have $\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial P_S(\alpha)} > 0$. Moreover, for the shadow values, we have

$$\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial P_S(\alpha)} = \int_{\mu_S(\alpha)}^{\mu^{\max}} \mu dG(\mu)$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial P_S(\alpha)} = \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\frac{[[1-G(\boldsymbol{\mu_S}(\alpha))]\,\boldsymbol{H_G} + [1-G(\boldsymbol{\mu_B}(\alpha))]\,\boldsymbol{H_B} + g(\boldsymbol{\mu_S}(\alpha))\,\boldsymbol{\mu_S}(\alpha)\,\boldsymbol{H_G}]\,P_S(\alpha)}{W_1 + H_B P_S(\alpha) + \overline{T}_1(\alpha)} + G(\boldsymbol{\mu_B}(\alpha))\,\boldsymbol{\mu_B}(\alpha) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu dG(\mu)$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial \overline{T}_1(\alpha)} = 0$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial \overline{T}_1(\alpha)} = \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\frac{[1-G(\boldsymbol{\mu_B}(\alpha))]\,P_S(\alpha)}{W + \boldsymbol{H_B} P_S(\alpha) + \overline{T}_1(\alpha)}$$

Regime ii: $P_S(\alpha) > \frac{Z}{\mu^{\max}}$ and $P_S(\alpha) < \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$. For $\boldsymbol{\mathcal{U}_1}(\alpha)$, we have

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial P_S(\alpha)} = \int_{\frac{\alpha Z}{P_S(\alpha)}}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\,\boldsymbol{H_B} dG(\mu) + \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\,H_G dG(\mu) + \left(\frac{\alpha Z}{P_S(\alpha)} - \boldsymbol{\mu_B}(\alpha)\right) g\left(\frac{\alpha Z}{P_S(\alpha)}\right)\frac{\alpha Z}{P_S(\alpha)}\boldsymbol{H_B} + (\boldsymbol{\mu_S}(\alpha) - \boldsymbol{\mu_B}(\alpha))\,g(\boldsymbol{\mu_S}(\alpha))\,\boldsymbol{\mu_S}(\alpha)\,H_G$$

and

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial \overline{T}_1(\alpha)} = \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\,dG(\mu)$$

For transfers $\mathbb{T}_{\mathbf{2}}(\alpha)$, we have

$$\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial P_S(\alpha)} = \left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\right)[1-G(\boldsymbol{\mu_S}(\alpha))]\,H_G + \left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\right)\left[1-G\left(\frac{\alpha Z}{P_S(\alpha)}\right)\right]\boldsymbol{H_B} - \left(\boldsymbol{\mu_S}(\alpha) - \boldsymbol{\mu_B}(\alpha) - \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\right) g(\boldsymbol{\mu_S}(\alpha))\,\boldsymbol{\mu_S}(\alpha)\,H_G - \left(\frac{\alpha Z}{P_S(\alpha)} - \boldsymbol{\mu_B}(\alpha) - \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\right) g\left(\frac{\alpha Z}{P_S(\alpha)}\right)\frac{\alpha Z}{P_S(\alpha)}\boldsymbol{H_B}$$

and

$$\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial \overline{T}_1(\alpha)} = (1+\chi)\left[\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}[1-G(\boldsymbol{\mu_B}(\alpha))]\right]$$

Moreover, for the shadow values, we have

$$\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial P_S(\alpha)} = \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu dG(\mu)$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial P_S(\alpha)} = \int_{\frac{\alpha Z}{P_S(\alpha)}}^{\mu^{\max}} \mu dG(\mu)$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial \overline{T}_1(\alpha)} = \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial \overline{T}_1(\alpha)} = 0$$

Regime iii: $P_S(\alpha) > \frac{Z}{\mu^{\max}}$ and $P_S(\alpha) = \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$. This regime is different than the others as $P_S(\alpha)$ cannot be chosen independently from $\overline{T}_1(\alpha)$ because of the constraint $P_S(\alpha) = \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$. However, ti is possible to write the optimality conditions in this regime in terms of derivatives that connect to the other regimes. For $\boldsymbol{\mathcal{U}_1}(\alpha)$, we have

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial P_S(\alpha)} = \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha)) \boldsymbol{H_B} dG(\mu) + \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha)) H_G dG(\mu) + \\ \boldsymbol{\mu_S}(\alpha)(\boldsymbol{\mu_S}(\alpha) - \boldsymbol{\mu_B}(\alpha)) H_G g(\boldsymbol{\mu_S}(\alpha)) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha)) dG(\mu) \frac{\partial \overline{T}_1(\alpha)}{\partial P_S(\alpha)}$$

$$\frac{\partial \boldsymbol{\mu_B}(\alpha)}{\partial P_S(\alpha)} = \frac{[1 - G(\boldsymbol{\mu_B}(\alpha))] \boldsymbol{H_B} + g(\boldsymbol{\mu_S}(\alpha)) \boldsymbol{\mu_S}(\alpha) H_G + [1 - G(\boldsymbol{\mu_S}(\alpha))] H_G + [1 - G(\boldsymbol{\mu_B}(\alpha))] \frac{\partial \overline{T}_1(\alpha)}{\partial P_S(\alpha)}}{g(\boldsymbol{\mu_B}(\alpha)) \left[W_1 + \boldsymbol{H_B} P_S(\alpha) + \overline{T}_1(\alpha)\right]}$$

and for $\mathbb{T}_{\boldsymbol{2}}(\alpha)$, we have

$$\frac{\partial \mathbb{T}_{\boldsymbol{2}}(\alpha)}{\partial P_S(\alpha)} = (1+\chi) \left[\left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))} \right) [[1 - G(\boldsymbol{\mu_S}(\alpha))] + g(\boldsymbol{\mu_S}(\alpha)) \boldsymbol{\mu_S}(\alpha)] \boldsymbol{H_G} + \right. \\ \left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))} [1 - G(\boldsymbol{\mu_B}(\alpha))] \right) \boldsymbol{H_B} - \boldsymbol{\mu_S}(\alpha) g(\boldsymbol{\mu_S}(\alpha)) \boldsymbol{\mu_S}(\alpha) \boldsymbol{H_G} + \\ \left. \left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))} [1 - G(\boldsymbol{\mu_B}(\alpha))] \right) \frac{\partial \overline{T}_1(\alpha)}{\partial P_S(\alpha)} \right]$$

where

$$\frac{\partial \overline{T}_1(\alpha)}{\partial P_S(\alpha)} = \frac{1}{\left[1 - G\left(\frac{\alpha Z}{P_S(\alpha)}\right)\right] P_S(\alpha)} \left[\left[1 - G\left(\frac{\alpha Z}{P_S(\alpha)}\right)\right] \boldsymbol{H_B} P_S(\alpha) + \left[1 - G\left(\frac{Z}{P_S(\alpha)}\right)\right] H_G P_S(\alpha) + \right. \\ \left. \left[W_1 + \boldsymbol{H_B} P_S(\alpha) + \overline{T}_1(\alpha)\right] \frac{\alpha Z}{P_S(\alpha)} g\left(\frac{\alpha Z}{P_S(\alpha)}\right) + H_G P_S(\alpha) \frac{Z}{P_S(\alpha)} g\left(\frac{Z}{P_S(\alpha)}\right) \right]$$

Moreover, for the shadow values, we have

$$\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial P_S(\alpha)} = \int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu dG(\mu)$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial P_S(\alpha)} = \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))} \frac{[[1 - G(\boldsymbol{\mu_S}(\alpha))] H_G + [1 - G(\boldsymbol{\mu_B}(\alpha))] \boldsymbol{H_B} + g(\boldsymbol{\mu_S}(\alpha)) \boldsymbol{\mu_S}(\alpha) H_G] P_S(\alpha)}{W_1 + \boldsymbol{H_B} P_S(\alpha) + \overline{T}_1(\alpha)} + \\ G(\boldsymbol{\mu_B}(\alpha)) \boldsymbol{\mu_B}(\alpha) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu dG(\mu) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))} \frac{[1 - G(\boldsymbol{\mu_B}(\alpha))] P_S(\alpha)}{W_1 + \boldsymbol{H_B} P_S(\alpha) + \overline{T}_1(\alpha)} \frac{\partial \overline{T}_1(\alpha)}{\partial P_S(\alpha)}$$

Note that the optimality conditions in Regimes *i* and *ii* limit to the optimality condition in Regime *iii*.

Regime iv: $P_S(\alpha) < \frac{Z}{\mu^{\max}}$ and $P_S(\alpha) > \frac{\alpha Z}{\mu_B(\alpha)}$. We have

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial P_S(\alpha)} = \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha)) \boldsymbol{H_B} dG(\mu)$$

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial \overline{T}_1(\alpha)} = \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\, dG(\mu)$$

$$\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial P_S(\alpha)} = \left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\left[1 - G(\boldsymbol{\mu_B}(\alpha))\right]\right)\boldsymbol{H_B}$$

$$\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial P_S(\alpha)} = \boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\left[1 - G(\boldsymbol{\mu_B}(\alpha))\right]$$

and

$$\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial P_S(\alpha)} = \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial \overline{T}_1(\alpha)} = 0$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial P_S(\alpha)} = \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\frac{\left[1 - G(\boldsymbol{\mu_B}(\alpha))\right]\boldsymbol{H_B} P_S(\alpha)}{W + \boldsymbol{H_B} P_S(\alpha) + \overline{T}_1(\alpha)} + G(\boldsymbol{\mu_B}(\alpha))\,\boldsymbol{\mu_B}(\alpha) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu\, dG(\mu)$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial \overline{T}_1(\alpha)} = \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\frac{\left[1 - G(\boldsymbol{\mu_B}(\alpha))\right] P_S(\alpha)}{W + \boldsymbol{H_B} P_S(\alpha) + \overline{T}_1(\alpha)}$$

Regime v: $P_S(\alpha) < \frac{Z}{\mu^{\max}}$ and $P_S(\alpha) < \frac{\alpha Z}{\mu_B(\alpha)}$. We have

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial P_S(\alpha)} = \int_{\frac{\alpha Z}{P_S(\alpha)}}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\,\boldsymbol{H_B}\, dG(\mu) + \left(\frac{\alpha Z}{P_S(\alpha)} - \boldsymbol{\mu_B}(\alpha)\right)\frac{\alpha Z}{P_S(\alpha)}\boldsymbol{H_B}\, g\left(\frac{\alpha Z}{P_S(\alpha)}\right)$$

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial \overline{T}_1(\alpha)} = \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\, dG(\mu)$$

$$\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial P_S(\alpha)} = \left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\right)\left[1 - G\left(\frac{\alpha Z}{P_S(\alpha)}\right) + g\left(\frac{\alpha Z}{P_S(\alpha)}\right)\frac{\alpha Z}{P_S(\alpha)}\right]\boldsymbol{H_B} - \frac{\alpha Z}{P_S(\alpha)}\, g\left(\frac{\alpha Z}{P_S(\alpha)}\right)\frac{\alpha Z}{P_S(\alpha)}\boldsymbol{H_B}$$

$$\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial \overline{T}_1(\alpha)} = \boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\left[1 - G(\boldsymbol{\mu_B}(\alpha))\right]$$

and

$$\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial P_S(\alpha)} = \frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial \overline{T}_1(\alpha)} = \frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial \overline{T}_1(\alpha)} = 0$$

$$\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial P_S(\alpha)} = \int_{\frac{\alpha Z}{P_S(\alpha)}}^{\mu^{\max}} \mu\, dG(\mu)$$

Regime vi: $P_S(\alpha) < \frac{Z}{\mu^{\max}}$ and $P_S(\alpha) = \frac{\alpha Z}{\mu_B(\alpha)}$. We have

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial P_S(\alpha)} = \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\,\boldsymbol{H_B}\, dG(\mu) + \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\, dG(\mu)\,\frac{\partial \overline{T}_1(\alpha)}{\partial P_S(\alpha)}$$

$$\frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial P_S(\alpha)} = \left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\left[1 - G(\boldsymbol{\mu_B}(\alpha))\right]\right)\boldsymbol{H_B} + \left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}\left[1 - G(\boldsymbol{\mu_B}(\alpha))\right]\right)\frac{\partial \overline{T}_1(\alpha)}{\partial P_S(\alpha)}$$

where

$$\frac{\partial \overline{T}_1(\alpha)}{\partial P_S(\alpha)} = -\frac{\left[1 - G\left(\frac{\alpha Z}{P_S(\alpha)}\right)\right]\boldsymbol{H_B} P_S(\alpha) + \left[W_1 + \boldsymbol{H_B} P_S(\alpha) + \overline{T}_1(\alpha)\right]\frac{\alpha Z}{P_S(\alpha)}\, g\left(\frac{\alpha Z}{P_S(\alpha)}\right)}{\left[1 - G\left(\frac{\alpha Z}{P_S(\alpha)}\right)\right] P_S(\alpha)}$$

Note that the optimality conditions in Regimes *iv* and *v* limit to the optimality condition in Regime *vi*.

Next, I present a series of lemmas that uncover some properties of the different regimes. Start with normal states.

Lemma 5. *In Regime i, $P_S(\alpha)$ and $\overline{T}_1(\alpha)$ are independent of α.*

Proof. Immediate from the fact that the optimality conditions in Regime i are independent of α. $\square$

Lemma 6. *The optimal $\boldsymbol{\mu_B}(\alpha)$ in Regime i is smaller than the optimal $\boldsymbol{\mu_B}(\alpha)$ in Regime ii.*

Proof. The only difference between the optimality condition for $\overline{T}_1(\alpha)$ in Regimes i and ii is that the incentives effect is higher in Regime i. Optimality requires that the optimality condition for $\overline{T}_1(\alpha)$ is decreasing in $\overline{T}_1(\alpha)$, hence $\boldsymbol{\mu_B}(\alpha)$ is lower in Regime i. $\square$

Thus, there exists $\tilde{P}_S(\alpha), \tilde{\tilde{P}}_S(\alpha)$with $\tilde{P}_S(\alpha) < \tilde{\tilde{P}}_S(\alpha)$ such that if $P_S(\alpha) < \tilde{P}_S(\alpha)$, then $P_S(\alpha) < \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$ (which corresponds to Regime ii) and if $P_S(\alpha) > \tilde{\tilde{P}}_S(\alpha)$, then $P_S(\alpha) > \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$ (which corresponds to Regime i).

Lemma 7. *The optimality condition for $P_S(\alpha)$ in Regime ii at $P_S(\alpha) \to \tilde{P}_S(\alpha)$ is strictly greater than the optimality condition for $P_S(\alpha)$ in Regime i at $P_S(\alpha) \to \tilde{\tilde{P}}_S(\alpha)$, conditional on choosing $\overline{T}_1(\alpha)$ optimally.*

Proof. In the limits, and after replacing the optimality conditions of $\overline{T}_1(\alpha)$ into the optimality conditions for $P_S(\alpha)$, we get

$$\left[\frac{\int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu dG(\mu)}{1-G(\boldsymbol{\mu_S}(\alpha))} - \frac{\int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu dG(\mu)}{1-G(\boldsymbol{\mu_B}(\alpha))}\right]\boldsymbol{H_G} + \left[\boldsymbol{\mu_S}(\alpha) - \frac{\int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu dG(\mu)}{1-G(\boldsymbol{\mu_B}(\alpha))}\right]\frac{g(\boldsymbol{\mu_S}(\alpha))\boldsymbol{\mu_S}(\alpha)}{1-G(\boldsymbol{\mu_S}(\alpha))}H_G +$$
$$\chi\left[\boldsymbol{\mu_B}(\alpha)\frac{G(\boldsymbol{\mu_B}(\alpha))}{1-G(\boldsymbol{\mu_B}(\alpha))}\boldsymbol{H_G} + \left(\boldsymbol{\mu_S}(\alpha) + \boldsymbol{\mu_B}(\alpha)\frac{G(\boldsymbol{\mu_B}(\alpha))}{1-G(\boldsymbol{\mu_B}(\alpha))}\right)\frac{g(\boldsymbol{\mu_S}(\alpha))\boldsymbol{\mu_S}(\alpha)}{1-G(\boldsymbol{\mu_S}(\alpha))}\boldsymbol{H_G} +\right] +$$
$$\tilde{\Omega}\frac{\int_{\mu_S(\alpha)}^{\mu^{\max}} \mu dG(\mu)\frac{1}{C'(\boldsymbol{H_G})} - G(\boldsymbol{\mu_B}(\alpha))\boldsymbol{\mu_B}(\alpha) - \int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu dG(\mu)}{1-G(\boldsymbol{\mu_S}(\alpha))}$$

in Regime i, and

$$\left[\frac{\int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu dG(\mu)}{1-G(\boldsymbol{\mu_S}(\alpha))} - \frac{\int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu dG(\mu)}{1-G(\boldsymbol{\mu_B}(\alpha))}\right]\boldsymbol{H_G} + \left[\boldsymbol{\mu_S}(\alpha) - \frac{\int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu dG(\mu)}{1-G(\boldsymbol{\mu_B}(\alpha))}\right]\frac{g(\boldsymbol{\mu_S}(\alpha))\boldsymbol{\mu_S}(\alpha)}{1-G(\boldsymbol{\mu_S}(\alpha))}H_G +$$
$$\chi\left[\boldsymbol{\mu_B}(\alpha)\frac{G(\boldsymbol{\mu_B}(\alpha))}{1-G(\boldsymbol{\mu_B}(\alpha))}\boldsymbol{H_G} + \left(\boldsymbol{\mu_S}(\alpha) + \boldsymbol{\mu_B}(\alpha)\frac{G(\boldsymbol{\mu_B}(\alpha))}{1-G(\boldsymbol{\mu_B}(\alpha))}\right)\frac{g(\boldsymbol{\mu_S}(\alpha))\boldsymbol{\mu_S}(\alpha)}{1-G(\boldsymbol{\mu_S}(\alpha))}\boldsymbol{H_G}\right] +$$
$$\tilde{\Omega}\frac{\left[\int_{\boldsymbol{\mu_S}(\alpha)}^{\mu^{\max}} \mu dG(\mu)\frac{1}{C'(\boldsymbol{H_G})} - \int_{\frac{\alpha Z}{P_S(\alpha)}}^{\mu^{\max}} \mu dG(\mu)\right]}{1-G(\boldsymbol{\mu_S}(\alpha))}$$

in Regime ii. There are two differences between these two equations. First, the optimality condition in Regime i has an extra negative term, $-\tilde{\Omega}\frac{G(\boldsymbol{\mu_B}(\alpha))\boldsymbol{\mu_B}(\alpha)}{1-G(\boldsymbol{\mu_S}(\alpha))}$. Second, they are evaluated at different levels of $\boldsymbol{\mu_B}(\alpha)$ and $P_S(\alpha)$. In particular, $\boldsymbol{\mu_B}(\alpha)$ is higher in Regime ii and $P_S(\alpha)$ is higher in Regime i. Then, at the boundaries of each regime, the optimality condition in Regime ii is higher than in Regime i. $\square$

Lemma 8. *As a function of $P_S(\alpha)$, the optimality condition in Regime iii is higher than the optimality condition in Regime ii and lower than in Regime i when they overlap.*

Proof. The optimality conditions coincide at the boundaries. Moreover, the problem in Regime iii coincides with that of Regimes i and ii with the aditional constraint that $P_S(\alpha) = \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$. Thus, conditional on $P_S(\alpha)$, Regime iii is dominated when the other regimes exist. This implies the desired inequalities.

Lemmas 4 and 5 implies that the optimality condition in normal states is decreasing in $P_S(\alpha)$, so there is always a unique optimal solution conditional on no market collapse. In particular, the optimal policy is in Regime i when the solution is such that $P_S(\alpha) \geq \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$, it is in Regime ii when the solution is such that $P_S(\alpha) < \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$, and it is in Regime iii when a solution in Regimes i and ii don't exist. $\square$

Consider now the crisis states.

Lemma 9. *Regime iv is never optimal.*

Proof. In this regime, we have

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial P_S(\alpha)} - \frac{\chi}{1+\chi}\frac{\partial \mathbb{T}_2(\alpha)}{\partial P_S(\alpha)} = \left(\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial \overline{T}_1(\alpha)} - \frac{\chi}{1+\chi}\frac{\partial \mathbb{T}_2(\alpha)}{\partial \overline{T}_1(\alpha)}\right)\boldsymbol{H_B}$$

while

$$\frac{\partial \boldsymbol{H_G}}{\partial P_S(\alpha)} < \frac{\partial \boldsymbol{H_G}}{\partial \overline{T}_1(\alpha)}\boldsymbol{H_B}$$

Thus, transfers dominate a positive market wedge. □

Lemma 10. *The optimality condition for $P_S(\alpha)$ in Regimes v and vi coincide at $P_S(\alpha) \to \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$, conditional on choosing $\overline{T}_1(\alpha)$ optimally. The optimality condition is negative at this point.*

Proof. That they coincide is immediate from the previous derivations. The optimality conditions are

$$\int_{\frac{\alpha Z}{P_S(\alpha)}}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha))\boldsymbol{H_B} dG(\mu) \; - \; \chi\left[\left(\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}[1 - G(\boldsymbol{\mu_B}(\alpha))]\right)\boldsymbol{H_B}\right] \; - \; \tilde{\Omega}\int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} \mu dG(\mu) \quad < \quad 0$$

$$\int_{\boldsymbol{\mu_B}(\alpha)}^{\mu^{\max}} (\mu - \boldsymbol{\mu_B}(\alpha)) dG(\mu) - \chi\left[\boldsymbol{\mu_B}(\alpha) + \frac{G(\boldsymbol{\mu_B}(\alpha))}{g(\boldsymbol{\mu_B}(\alpha))}[1 - G(\boldsymbol{\mu_B}(\alpha))]\right] = 0$$

since $\tilde{\Omega} > 0$. □

Lemma 11. *Regime vi is dominated by Regime v.*

Proof. Both regimes coincide at $P_S(\alpha) \to \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$. For lower levels of $P_S(\alpha)$, Regime *vi* solves the same problem with the additional constraint $P_S(\alpha) = \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$. Hence, Regime *vi* is dominated by Regime *v*. □

Finally,

Lemma 12. *There exists $0 \leq \tilde{\alpha}^* < \tilde{\alpha}^{**} \leq 1$ such that the optimal policy is in Regime i. If $\alpha < \tilde{\alpha}^*$, the economy is in Regime v.*

Proof. By Assumption 1, if $P_S(\alpha) \leq \frac{\alpha Z}{\boldsymbol{\mu_B}} < \alpha Z$, then $\lambda_E = 1$. Then, a solution with $\lambda_E \in (0,1)$ requires that $P_S(\alpha) > \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$ and Regime *i* is the only regime that satisfies this condition.

Now, let's consider the thresholds $\tilde{\alpha}$ that separates crisis and normal regimes. □

Lemma 13. *Conditional on $\tilde{\Omega}$, and if λ_E is sufficiently large, there is a unique threshold $\tilde{\alpha}$ such that for $\alpha < \tilde{\alpha}$ the economy is in crisis states, and if $\alpha > \tilde{\alpha}$, the economy is in normal states.*

Proof. The first-order necessary condition of $\boldsymbol{W}$ with respect to $\tilde{\alpha}$ is

$$\boldsymbol{W}'(\tilde{\alpha}) = \left[\boldsymbol{U_1}^C(\tilde{\alpha}^-) - \boldsymbol{U_1}^N(\tilde{\alpha}^+) - \chi\left[\mathbb{T}_2^C(\tilde{\alpha}^-) - \mathbb{T}_2^N(\tilde{\alpha}^+)\right]\right] f(\tilde{\alpha}) + \tilde{\Omega}\left[\frac{\partial \gamma_G}{\partial \tilde{\alpha}}\frac{1}{C'(H_G)} - \frac{\partial \gamma_B}{\partial \tilde{\alpha}}\right] = 0$$

We want If we show that $\boldsymbol{W}''(\alpha) < 0$, then there is only one $\tilde{\alpha}$ that satisfies this equation and is such that $\boldsymbol{W}'(\alpha) > 0$ for $\alpha < \tilde{\alpha}$ and $\boldsymbol{W}'(\alpha) < 0$ for $\alpha > \tilde{\alpha}$, which implies that the transition is from a crisis state to a normal state. We have

$$\boldsymbol{W}''(\alpha) = \left[\frac{\partial \boldsymbol{U_1}^C(\alpha)}{\partial \alpha} - \frac{\partial \boldsymbol{U_1}^N(\alpha)}{\partial \alpha} - \chi\left[\frac{\partial \mathbb{T}_2^C(\alpha)}{\partial \alpha} - \frac{\partial \mathbb{T}_2^N(\alpha)}{\partial \alpha}\right]\right] f(\tilde{\alpha}) + \tilde{\Omega}\left[\frac{\partial^2 \gamma_G}{\partial \tilde{\alpha}^2}\frac{1}{C'(H_G)} - \frac{\partial^2 \gamma_B}{\partial \tilde{\alpha}^2}\right]$$

Note that when $\alpha = \tilde{\alpha}^+$, the economy is in Regime *i*, so P_S , $\overline{T}_1$ and $\boldsymbol{\mu_B}$ are independent of α in a neighborhood of $\tilde{\alpha}$:

$$\boldsymbol{\mathcal{U}_1}(\tilde{\alpha}^+) - \chi\mathbb{T}_2(\tilde{\alpha}^+) = \int_{\boldsymbol{\mu_B}}^{\mu^{\max}} \mu[W_1 +] dG(\mu) + \int_{\boldsymbol{\mu_B}}^{\mu^{\max}} \mu P_S \boldsymbol{H_B} dG(\mu) + \int_{\boldsymbol{\mu_S}}^{\mu^{\max}} \mu P_S H_G dG(\mu) -$$
$$\chi\left[\left[\boldsymbol{\mu_B} P_S \boldsymbol{S} - \left[[1 - G(\boldsymbol{\mu_S})] H_G Z + \boldsymbol{H_B}\alpha Z\right]\right] + \boldsymbol{\mu_B}\overline{T}_1\right]$$

Hence

$$\frac{\partial \boldsymbol{\mathcal{U}_1}(\tilde{\alpha}^+) - \chi\mathbb{T}_2(\tilde{\alpha}^+)}{\partial \tilde{\alpha}} = \chi Z \boldsymbol{H_B} > 0$$

When $\alpha = \tilde{\alpha}^-$, the economy is in Regime v, so $\boldsymbol{\mu_B}$ is constant and $P_S(\alpha) = \frac{(1-\tau)\alpha Z}{\boldsymbol{\mu_B}}$, for some $\tau > 0$:

$$\boldsymbol{\mathcal{U}_1}\left(\tilde{\alpha}^-\right) - \chi \mathbb{T}_{\mathbf{2}}\left(\tilde{\alpha}^-\right) = \int_{\boldsymbol{\mu_B}}^{\mu^{\max}} \mu\left[W_1 + \overline{T}_1\left(\tilde{\alpha}^-\right)\right] dG(\mu) + \int_{\frac{\boldsymbol{\mu_B}}{1-\tau}}^{\mu^{\max}} \mu \frac{(1-\tau)\tilde{\alpha} Z}{\boldsymbol{\mu_B}} \boldsymbol{H_B} dG(\mu) - \chi\left[\boldsymbol{\mu_B}\left[\frac{(1-\tau)\alpha Z}{\boldsymbol{\mu_B}} - \boldsymbol{P_B}\left(\tilde{\alpha}^-\right)\right]\boldsymbol{S}\left(\tilde{\alpha}^-\right) + \boldsymbol{\mu_B}\overline{T}_1\left(\tilde{\alpha}^-\right)\right]$$

where

$$\overline{T}_1\left(\tilde{\alpha}^-\right) = \frac{G\left(\boldsymbol{\mu_B}^{R_5}\right) W_1 - \left[1 - G\left(\frac{\tilde{\alpha}^- Z}{P_S(\tilde{\alpha}^-)}\right)\right] \boldsymbol{H_B} \frac{(1-\omega)\tilde{\alpha}^- Z}{\boldsymbol{\mu_B}^{R_5}}}{1 - G\left(\boldsymbol{\mu_B}^{R_5}\right)}$$

Hence

$$\frac{\partial \boldsymbol{\mathcal{U}_1}\left(\tilde{\alpha}^-\right) - \chi \mathbb{T}_{\mathbf{2}}\left(\tilde{\alpha}^-\right)}{\partial \tilde{\alpha}} = \left[\frac{\int_{\frac{\tilde{\alpha}^- Z}{P_S(\tilde{\alpha}^-)}}^{\mu^{\max}} \mu dG(\mu)}{1 - G\left(\frac{\tilde{\alpha}^- Z}{P_S(\tilde{\alpha}^-)}\right)} - \frac{\int_{\boldsymbol{\mu_B}^{R_5}}^{\mu^{\max}} \mu dG(\mu)}{1 - G\left(\boldsymbol{\mu_B}^{R_5}\right)}\right]\left[1 - G\left(\frac{\tilde{\alpha}^- Z}{P_S\left(\tilde{\alpha}^-\right)}\right)\right] \boldsymbol{H_B} \frac{(1-\omega) Z}{\boldsymbol{\mu_B}^{R_5}} + \chi \frac{1 - \omega G\left(\boldsymbol{\mu_B}^{R_5}\right)}{1 - G\left(\boldsymbol{\mu_B}^{R_5}\right)} Z \left[1 - G\left(\frac{\tilde{\alpha}^- Z}{P_S\left(\tilde{\alpha}^-\right)}\right)\right] \boldsymbol{H_B} > 0$$

Note that these expressions are proportional to $\boldsymbol{H_B}$, so if λ_E is large, these effect is small (note that the difference in **levels** is big, it just doesn't change much with $\tilde{\alpha}$).

Moreover, we have

$$\frac{\partial \gamma_G}{\partial \tilde{\alpha}} = \left[\left[1 - G\left(\frac{Z}{P_S}\right)\right] Z - \int_{\frac{Z}{P_S}}^{\mu^{\max}} \mu P_S dG(\mu)\right] f(\tilde{\alpha})$$

Independent of $\tilde{\alpha}$ (except for $f(\tilde{\alpha})$), and

$$\frac{\partial \gamma_B}{\partial \tilde{\alpha}} = \left[G\left(\frac{\boldsymbol{\mu_B}}{1-\tau}\right)\tilde{\alpha} Z + \int_{\frac{\boldsymbol{\mu_B}}{1-\tau}}^{\mu^{\max}} \mu \frac{(1-\tau)\tilde{\alpha} Z}{\boldsymbol{\mu_B}} dG(\mu)\right] f(\tilde{\alpha}) - \left[\int_{\underline{\mu}}^{\mu_B} \mu_B P_S dG(\mu) + \int_{\mu_B}^{\mu^{\max}} \mu P_S dG(\mu)\right] f(\tilde{\alpha})$$

Then

$$\frac{\partial^2 \gamma_B}{\partial \tilde{\alpha}^2} = \left[G\left(\frac{\boldsymbol{\mu_B}}{1-\tau}\right) Z + \int_{\frac{\boldsymbol{\mu_B}}{1-\tau}}^{\mu^{\max}} \mu \frac{(1-\tau) Z}{\mu_B} dG(\mu)\right] f(\tilde{\alpha}) > 0$$

Hence, $W''(\alpha) < 0$. □

Lemma 14. *At $\tilde{\alpha}$, $P_S(\alpha)$ and $\overline{T}_1(\alpha)$ are discontinuous.*

Proof. In Regime v, $P_S(\alpha) < \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$, while in Regime i, $P_S(\alpha) > \frac{\alpha Z}{\boldsymbol{\mu_B}(\alpha)}$, and $\boldsymbol{\mu_B}(\alpha)$ is higher in Regime v. Hence, the price is higher in Regime i. Since $\boldsymbol{\mu_B}$ is higher in Regime v and the volume traded in the market for trees is lower, $\overline{T}_1(\alpha)$ is discontinuously higher in Regime v. □

Finally, recall that

$$\tilde{\Omega} = \frac{E\left[\frac{\partial \boldsymbol{\mathcal{U}_1}(\alpha)}{\partial H_G} + \left(1 - \alpha C'\left(\boldsymbol{H_G}\right)\right) Z + \frac{\chi}{1+\chi} \frac{\partial \mathbb{T}_{\mathbf{2}}(\alpha)}{\partial H_G}\right]}{\eta_G\left(\boldsymbol{H_G}\right) \boldsymbol{\gamma_B} + E\left[\frac{\partial \overline{\gamma}_{\boldsymbol{B}}(\alpha)}{\partial H_G}\right] - \frac{1}{C'(\boldsymbol{H_G})} E\left[\frac{\partial \overline{\gamma}_{\boldsymbol{G}}(\alpha)}{\partial H_G}\right]}.$$

Since $\boldsymbol{\mathcal{U}_1}(\alpha)$, $\mathbb{T}_{\mathbf{2}}(\alpha)$, $\overline{\gamma}_{\boldsymbol{G}}(\alpha)$, $\overline{\gamma}_{\boldsymbol{B}}(\alpha)$, and $\boldsymbol{H_G}$ functions of $\tilde{\Omega}$, the solution can be found as the fixed point implicitly defined by this equation. If more than one fixed point exists, we pick the one that generates the higher value for the planner.

□

Proof of Proposition 8. Consider first the case where the planner purchases a quantity $S_B(\alpha)$ of bad trees at the $P_S^{\omega}(\alpha)$. First, consider the market clearing of trees for the determination of $\mu_B(\alpha)$. On the one hand, in the economy with the transaction subsidy, the market clearing condition is

$$G(\mu_B(\alpha))\left[W_1 + H_B P_S^{\omega}(\alpha)\right] = \left[H_B + \left[1 - G(\mu_S(\alpha))\right] H_G\right] P_B(\alpha) + \left[1 - G(\mu_B(\alpha))\right] Q_1^B(\alpha) B(\alpha) \\ Q_1^B(\alpha)\left[B'(\alpha) - B(\alpha)\right]$$

Using that

$$\left[P_S^{\omega}(\alpha) - P_B(\alpha)\right] S(\alpha) = Q_1^B(\alpha)\left[B'(\alpha) - B(\alpha)\right],$$

we get

$$G(\mu_B(\alpha))W_1 = \left[\left[1 - G(\mu_B(\alpha))\right] H_B + \left[1 - G(\mu_S(\alpha))\right] H_G\right] P_S^{\omega}(\alpha) + \left[1 - G(\mu_B(\alpha))\right] Q_1^B(\alpha) B(\alpha).$$

On the other hand, in the economy with the asset purchase program, the market clearing condition is

$$\begin{aligned}G(\mu_B(\alpha))W_1 = \left[\left[1 - G(\mu_S(\alpha))\right] H_G + \left[1 - G(\mu_B(\alpha))\right] H_B\right] P_S^{\omega}(\alpha) - S_B(\alpha) P_S^{\omega}(\alpha) + \\ \left[1 - G(\mu_B(\alpha))\right] Q_1^B(\alpha) B(\alpha) + Q_1^B(\alpha)\left[B'(\alpha) - B(\alpha)\right]\end{aligned}$$

where I assumed the same price received by sellers in both economies. Using that

$$S_B(\alpha) P_S^{\omega}(\alpha) = Q_1^B(\alpha)\left[B'(\alpha) - B(\alpha)\right],$$

we get

$$G(\mu_B(\alpha))W_1 = \left[\left[1 - G(\mu_S(\alpha))\right] H_G + \left[1 - G(\mu_B(\alpha))\right] H_B\right] P_S^{\omega}(\alpha) + \left[1 - G(\mu_B(\alpha))\right] Q_1^B(\alpha) B(\alpha).$$

Note that, conditional on the price received by sellers, both economies feature the same $\mu_B(\alpha)$.

The two instruments generate the same price if and only if $P_M(\alpha) = P_S^{\omega}(\alpha)$, where $P_M(\alpha)$ is the price in the economy with the asset purchase program. We have

$$P_M(\alpha) = \frac{\frac{\left[1 - G\left(\frac{Z}{P_S^{\omega}(\alpha)}\right)\right] H_G}{\left[1 - G\left(\frac{Z}{P_S^{\omega}(\alpha)}\right)\right] H_G + H_B - S_B(\alpha)} Z + \frac{H_B - S_B(\alpha)}{\left[1 - G\left(\frac{Z}{P_S^{\omega}(\alpha)}\right)\right] H_G + H_B - S_B(\alpha)} \alpha Z}{\mu_B(\alpha)}.$$

Then, we need,

$$S_B(\alpha) = H_B - \frac{Z - \mu_B(\alpha) P_S^{\omega}(\alpha)}{\mu_B(\alpha) P_S^{\omega}(\alpha) - \alpha Z}\left[1 - G\left(\frac{Z}{P_S^{\omega}(\alpha)}\right)\right] H_G$$

Note that $S_B(\alpha) > 0$ if $\omega(\alpha) > 0$ and $\mu_B(\alpha) P_S^{\omega}(\alpha) < Z$.

Thus, we only need to show that the two instruments have the same cost for the planner. The cost of the transaction subsidy is

$$\begin{aligned}\mathbb{T}_2^{\omega}(\alpha) &= (1+\chi)\left[B(\alpha) + \mu_B(\alpha)\left[P_S^{\omega}(\alpha) - P_B(\alpha)\right] S(\alpha)\right] \\ &= (1+\chi)\left[B(\alpha) + \left[\mu_B(\alpha) P_S^{\omega}(\alpha) S(\alpha) - \left[\left[1 - G\left(\frac{Z}{P_S^{\omega}(\alpha)}\right)\right] H_G Z + \alpha Z H_B\right]\right]\right]\end{aligned}$$

while the cost of the asset purchase program is

$$\begin{aligned}\mathbb{T}_2^{AP}(\alpha) &= (1+\chi)\left[B(\alpha) + \mu_B(\alpha) P_S^{\omega}(\alpha) S_B(\alpha)\right] \\ &= (1+\chi)\left[B(\alpha) + \left[\mu_B(\alpha) P_S^{\omega}(\alpha) S(\alpha) - \mu_B(\alpha) P_S^{\omega}(\alpha) \frac{Z - \alpha Z}{\mu_B(\alpha) P_S^{\omega}(\alpha) - \alpha Z}\left[1 - G\left(\frac{Z}{P_S^{\omega}(\alpha)}\right)\right] H_G\right]\right] \\ &= (1+\chi)\left[B(\alpha) + \left[\mu_B(\alpha) P_S^{\omega}(\alpha) S(\alpha) - \left[\lambda_M(\alpha) Z + (1 - \lambda_M(\alpha))\alpha Z\right] \frac{\left[1 - G\left(\frac{Z}{P_S^{\omega}(\alpha)}\right)\right] H_G}{\lambda_M(\alpha)}\right]\right] \\ &= (1+\chi)\left[B(\alpha) + \left[\mu_B(\alpha) P_S^{\omega}(\alpha) S(\alpha) - \left[\left[1 - G\left(\frac{Z}{P_S^{\omega}(\alpha)}\right)\right] H_G Z + \alpha Z H_B\right]\right]\right]\end{aligned}$$

Thus, both instruments have the same cost.

Note that there are no equilibria where the market price is less than $P_S^{\omega}(\alpha)$ as then agents would only sell to the government. Also, there is no equilibrium where the market price is higher than $P_S^{\omega}(\alpha)$, as in that case no agent would sell to the government, but $P_S^{\omega}(\alpha)$ is higher than *laissez-faire* by assumption. Hence, either the market price is $P_S^{\omega}(\alpha)$ or the government buys all the trees. In all cases, the net cost for the government is the same in all equilibria, as any additional purchases are fairly priced.

□

Proof of Proposition 9. First, note that WLOG we can express $c_1(x;\alpha)$ as $c_1(x;\alpha) = T_1(\mu;\alpha) + \Omega(\mu,\omega_1;\alpha) + \Xi(x;\alpha)$, with $\Omega(\mu,0;\alpha) = 0$ and $\Xi(\mu,\omega_1,0,0;\alpha) = 0$. I divide the rest of the proof into a series of lemmas.

Lemma 15. *$c_1(x;\alpha)$ is weakly increasing in μ and $c_2(x;\alpha)$ is weakly decreasing in μ.*

Proof. Fix (ω_1, h_G, h_B). Take $\mu'' > \mu'$. Then, the ICs imply

$$\mu''\left[c_1\left(\mu'', \omega_1, h_G, h_B; \alpha\right) - c_1\left(\mu', \omega_1, h_G, h_B; \alpha\right)\right] \geq c_2\left(\mu', \omega_1, h_G, h_B; \alpha\right) - c_2\left(\mu'', \omega_1, h_G, h_B; \alpha\right),$$

and

$$\mu'\left[c_1\left(\mu', \omega_1, h_G, h_B; \alpha\right) - c_1\left(\mu'', \omega_1, h_G, h_B; \alpha\right)\right] \geq c_2\left(\mu'', \omega_1, h_G, h_B; \alpha\right) - c_2\left(\mu', \omega_1, h_G, h_B; \alpha\right).$$

Adding up the two constraints, we get

$$\left(\mu'' - \mu'\right)\left[c_1\left(\mu'', \omega_1, h_G, h_B; \alpha\right) - c_1\left(\mu', \omega_1, h_G, h_B; \alpha\right)\right] \geq 0,$$

which implies that $c_1(\mu'', \omega_1, h_G, h_B; \alpha) \geq c_1(\mu', \omega_1, h_G, h_B; \alpha)$ and $c_2(\mu', \omega_1, h_G, h_B; \alpha) \geq c_2(\mu'', \omega_1, h_G, h_B; \alpha) - \mu'[c_1(\mu', \omega_1, h_G, h_B; \alpha) - c_1(\mu'', \omega_1, h_G, h_B; \alpha)] \geq c_2(\mu'', \omega_1, h_G, h_B; \alpha)$. Moreover, since c_1 and c_2 are monotonic in μ, they are differentiable a.e. with respect to μ. ☐

Lemma 16. *Suppose $T_1(\mu;\alpha) > 0$ for some $\mu \in (1, \mu^{\max}]$. Then, there exists $\mu_B(\alpha) \in [1, \mu']: T_1(\mu;\alpha) > 0 \iff \mu \geq \mu_B(\alpha)$. Moreover, $\forall \mu', \mu'' \geq \mu_B(\alpha)$, $T_1(\mu';\alpha) = T_1(\mu'';\alpha) \equiv \overline{T}_1(\alpha)$.*

Proof. Fix $\omega_1 = h_G = h_B = 0$. To simplify notation, let $x_0 \equiv (\mu, 0, 0, 0; \alpha)$.Then, Lemma 1 implies that $T_1(\mu;\alpha)$ is weakly increasing in $\mu$and, therefore, $T_1(\mu;\alpha) > 0$ for all $\mu' > \mu$. Let $\mu_B(\alpha) = \inf_{\mu \in [1, \mu^{\max}]} \mu$ such that $T_1(\mu;\alpha) > 0$.

The IC constraints with a continuum of types require that $\mu \frac{\partial T_1(\mu;\alpha)}{\partial \mu} + \frac{\partial c_2(x_0;\alpha)}{\partial \mu} = 0$ where T_1 and c_2 are differentiable with respect to μ, and $\lim_{\mu \to \tilde{\mu}^-} \mu T_1(\mu;\alpha) + c_2(x_0;\alpha) = \lim_{\mu \to \tilde{\mu}^+} \mu T_1(\mu;\alpha) + c_2(x_0;\alpha)$ if T_1 or c_2 are discontinuous at $\tilde{\mu}$.[37] Suppose not, so $T_1(\mu;\alpha)$ is strictly increasing in μ at some μ'. Consider first the case where T_1 and c_2 are differentiable at μ'. We will find an alternative plan that reduces the consumption of types below μ' and increase the consumption of types above μ'. This will increase the planner's objective and reduce costs. Take $\underline{\mu}, \overline{\mu}: \mu_B(\alpha) < \underline{\mu} < \mu' < \overline{\mu}$, which implies that $T_1\left(\underline{\mu};\alpha\right) < T_1(\mu';\alpha) < T_1(\overline{\mu};\alpha)$, and $\epsilon, \delta > 0$ such that

$$\int_{\underline{\mu}}^{\mu'} \epsilon\left[T_1(\mu;\alpha) - T_1\left(\underline{\mu};\alpha\right)\right]\left[T_1(\mu';\alpha) - T_1(\mu;\alpha)\right] dG(\mu) =$$
$$\int_{\mu'}^{\overline{\mu}} \delta\left[T_1(\mu;\alpha) - T_1\left(\mu';\alpha\right)\right]\left[T_1(\overline{\mu};\alpha) - T_1(\mu;\alpha)\right] dG(\mu)$$

or

$$\delta = \frac{\int_{\underline{\mu}}^{\mu'}\left[T_1(\mu;\alpha) - T_1\left(\underline{\mu};\alpha\right)\right]\left[T_1(\mu';\alpha) - T_1(\mu;\alpha)\right] dG(\mu)}{\int_{\mu'}^{\overline{\mu}}\left[T_1(\mu;\alpha) - T_1(\mu';\alpha)\right]\left[T_1(\overline{\mu};\alpha) - T_1(\mu;\alpha)\right] dG(\mu)}\epsilon$$

We will consider the limit $\epsilon \to 0$ and $\overline{\mu} \to \mu'$ with $\delta > 0$ and finite. Consider the following alternative allocation:

$$\hat{T}_1(\mu;\alpha) = \begin{cases} T_1(\mu;\alpha) & \text{if } \mu \leq \underline{\mu} \\ T_1(\mu;\alpha) - \epsilon\left[T_1(\mu;\alpha) - T_1\left(\underline{\mu};\alpha\right)\right]\left[T_1(\mu';\alpha) - T_1(\mu;\alpha)\right] & \text{if } \mu \in \left(\underline{\mu}, \mu'\right] \\ T_1(\mu;\alpha) + \delta\left[T_1(\mu;\alpha) - T_1(\mu';\alpha)\right]\left[T_1(\overline{\mu};\alpha) - T_1(\mu;\alpha)\right] & \text{if } \mu \in (\mu', \overline{\mu}] \\ T_1(\mu;\alpha) & \text{if } \mu > \overline{\mu} \end{cases}$$

and

$$\hat{c}_2(x_0;\alpha) = \begin{cases} c_2(x_0;\alpha) - \Delta & \text{if } \mu \leq \underline{\mu} \\ c_2(x_0;\alpha) + \mu\epsilon\left[T_1(\mu;\alpha) - T_1\left(\underline{\mu};\alpha\right)\right]\left[T_1(\mu';\alpha) - T_1(\mu;\alpha)\right] - \\ \quad \epsilon\int_{\underline{\mu}}^{\mu}\left[T_1(\tilde{\mu};\alpha) - T_1\left(\underline{\mu};\alpha\right)\right]\left[T_1(\mu';\alpha) - T_1(\tilde{\mu};\alpha)\right] d\tilde{\mu} - \Delta & \text{if } \mu \in \left(\underline{\mu}, \mu'\right] \\ c_2(x_0;\alpha) - \mu\delta\left[T_1(\mu;\alpha) - T_1\left(\mu';\alpha\right)\right]\left[T_1(\overline{\mu};\alpha) - T_1(\mu;\alpha)\right] - \\ \quad \delta\int_{\mu}^{\overline{\mu}}\left[T_1(\tilde{\mu};\alpha) - T_1\left(\mu';\alpha\right)\right]\left[T_1(\overline{\mu};\alpha) - T_1(\tilde{\mu};\alpha)\right] d\tilde{\mu} & \text{if } \mu \in (\mu', \overline{\mu}] \\ c_2(x_0;\alpha) & \text{if } \mu > \overline{\mu} \end{cases}$$

[37] We cannot have that T_1 or c_2 are continuous at $\tilde{\mu}$ but not differentiable, as the requirement is that both directional derivatives are 0, which implies that T_1 and c_2 are differentiable at $\tilde{\mu}$.

where $\Delta \equiv \delta \int_{\mu'}^{\overline{\mu}} \left[T_1(\tilde{\mu};\alpha) - T_1(\mu';\alpha)\right]\left[T_1(\overline{\mu};\alpha) - T_1(\tilde{\mu};\alpha)\right] d\tilde{\mu} -$
$\epsilon \int_{\underline{\mu}}^{\mu'} \left[T_1(\tilde{\mu};\alpha) - T_1\left(\underline{\mu};\alpha\right)\right]\left[T_1(\mu';\alpha) - T_1(\tilde{\mu};\alpha)\right] d\tilde{\mu}$. It is straightforward to see that $\hat{T}_1$ and $\hat{c}_2$ are continuous for all $\mu \in \left(\underline{\mu}, \overline{\mu}\right)$ and satisfy the IC constraints. Moreover, the alternative allocation increases the planner's objective since it redistributes towards agents with higher μ, and in the limit as $\epsilon \to 0$ and $\overline{\mu} \to \mu'$ with $\delta > 0$ and finite, it require lower transfers in period 2:

$$\hat{c}_2(x_0;\alpha) = \begin{cases} c_2(x_0;\alpha) & \text{if } \mu \leq \underline{\mu} \\ c_2(x_0;\alpha) & \text{if } \mu \in \left(\underline{\mu}, \mu'\right] \\ c_2(x_0;\alpha) - \mu\delta\left[T_1(\mu;\alpha) - T_1(\mu';\alpha)\right]\left[T_1(\overline{\mu};\alpha) - T_1(\mu;\alpha)\right] & \text{if } \mu \in (\mu', \overline{\mu}] \\ c_2(x_0;\alpha) & \text{if } \mu > \overline{\mu} \end{cases}$$

which implies that the planner can choose $\hat{T}_2(x_0;\alpha) < T_2(x_0;\alpha)$.

Now suppose $T_1(\mu;\alpha)$ is discontinuous at μ', and $\lim_{\mu\to\mu'^-} T_1(\mu;\alpha) < T_1(\mu';\alpha)$. [38]We will find an alternative plan that increases the consumption of types in $[\hat{\mu}, \mu')$ and reduce the consumption of the types in $[\mu_B(\alpha), \hat{\mu})$, for some $\hat{\mu} \in (\mu_B(\alpha), \mu')$. This will increase the planner's objective and reduce costs. Fix $\hat{\mu}$ and $(\epsilon, \delta) > 0$ such that

$$\int_{\mu_B(\alpha)}^{\hat{\mu}} \epsilon dG(\mu) = \int_{\hat{\mu}}^{\mu'} \delta dG(\mu) \implies \delta = \frac{G(\hat{\mu}) - G(\mu_B(\alpha))}{G(\mu') - G(\hat{\mu})}\epsilon.$$

We will take the limit $\epsilon \to 0$ and $\hat{\mu} \to \mu'$ such that $\delta > 0$ and finite. Consider the following alternative allocation:

$$\hat{T}_1(\mu;\alpha) = \begin{cases} 0 & if\ \mu < \mu_B(\alpha) \\ T_1(\mu;\alpha) - \epsilon & if\ \mu \in [\mu_B(\alpha), \hat{\mu}) \\ T_1(\mu;\alpha) + \delta & if\ \mu \in [\hat{\mu}, \mu') \\ T_1(\mu;\alpha) & if\ \mu \geq \mu' \end{cases}$$

$$\hat{c}_2(x_0;\alpha) = \begin{cases} c_2(x_0;\alpha) + [\hat{\mu} - \mu_B(\alpha)]\epsilon - (\mu' - \hat{\mu})\delta & if\ \mu < \mu_B(\alpha) \\ c_2(x_0;\alpha) + \hat{\mu}\epsilon - (\mu' - \hat{\mu})\delta & if\ \mu \in [\mu_B(\alpha), \hat{\mu}) \\ c_2(x_0;\alpha) - \mu'\delta & if\ \mu \in [\hat{\mu}, \mu') \\ c_2(x_0;\alpha) & if\ \mu \geq \mu' \end{cases}$$

For ϵ and δ sufficiently small, $\hat{T}_1(\mu;\alpha), \hat{c}_2(x_0;\alpha) \geq 0$ for all μ and $\lim_{\mu\to\mu'^-} T_1(\mu;\alpha) \leq \lim_{\mu\to\mu'^+} T_1(\mu;\alpha)$. We need to check that the IC constraints are satisfied. It is immediate that in the intervals $(1, \mu_B(\alpha))$, $(\mu_B(\alpha), \hat{\mu})$, $(\hat{\mu}, \mu')$ and $(\hat{\mu}, \mu^{\max}]$, the IC constraints don't change as the alternative allocation only adds a constant to each interval. We need to check that the ICs are satisfied at $\mu \in [\mu_B(\alpha), \hat{\mu}, \mu']$. Since $\hat{T}_1(\mu;\alpha)$ and $\hat{c}_2(x_0;\alpha)$ are discontinuous at these values, we need to check that the agents utility from truthfull revelation is continuous. For $\mu = \mu_B(\alpha)$, we need

$$\lim_{\mu\to\mu_B(\alpha)^-} c_2(x_0;\alpha) + [\hat{\mu} - \mu_B(\alpha)]\epsilon - \left(\mu' - \hat{\mu}\right)\delta = \lim_{\mu\to\mu_B(\alpha)^+} \mu\left[T_1(\mu;\alpha) - \epsilon\right] + c_2(x_0;\alpha) + \hat{\mu}\epsilon - \left(\mu' - \hat{\mu}\right)\delta$$

Noting that $\lim_{\mu\to\mu_B(\alpha)^-} c_2(x_0;\alpha) = \lim_{\mu\to\mu_B(\alpha)^+} \mu T_1(\mu;\alpha) + c_2(x_0;\alpha)$, it is immediate that the equality holds. For $\mu = \hat{\mu}$, we need

$$\lim_{\mu\to\hat{\mu}^-} \mu\left[T_1(\mu;\alpha) - \epsilon\right] + c_2(x_0;\alpha) + \hat{\mu}\epsilon - \left(\mu' - \hat{\mu}\right)\delta = \lim_{\mu\to\hat{\mu}^+} \mu\left[T_1(\mu;\alpha) + \delta\right] + c_2(x_0;\alpha) - \mu'\delta$$

Noting that $\lim_{\mu\to\hat{\mu}^-} \mu T_1(\mu;\alpha) + c_2(x_0;\alpha) = \lim_{\mu\to\hat{\mu}^+} \mu T_1(\mu;\alpha) + c_2(x_0;\alpha)$, it is immediate that the equality holds. For $\mu = \hat{\mu}$, we need

$$\lim_{\mu\to\mu'^-} \mu\left[T_1(\mu;\alpha) + \delta\right] + c_2(x_0;\alpha) - \mu'\delta = \lim_{\mu\to\mu'^+} \mu T_1(\mu;\alpha) + c_2(x_0;\alpha)$$

Noting that $\lim_{\mu\to\mu'^-} \mu T_1(\mu;\alpha) + c_2(x_0;\alpha) = \lim_{\mu\to\mu'^+} \mu T_1(\mu;\alpha) + c_2(x_0;\alpha)$, it is immediate that the equality holds. Finally, the alternative allocation increases the planner's objective since it redistributes towards agents with higher μ, and it require lower transfers in period 2 when $\epsilon \to 0$ and $\hat{\mu} \to \mu'$ with $\delta > 0$ and finite, so the planner can choose $\hat{T}_2(x_0;\alpha) > T_2(x_0;\alpha)$.

Finally, note that the alternative allocations do not affect the incentives to produce trees in period 0. ☐

[38]The proof with $T_1(\mu';\alpha) < \lim_{\mu\to\mu'^+} T_1(\mu';\alpha)$ is analogous.

Lemma 17. *Suppose* $\Omega(\mu,\omega_1;\alpha) > 0$ *for some* $\mu \in (1,\mu^{\max}]$*. Then,* $\Omega(\mu,\omega_1;\alpha) > 0 \iff \mu \geq \mu_B(\alpha)$*. Moreover, if* $\mu \geq \mu_B(\alpha)$*,* $\Omega(\mu,\omega_1;\alpha) = \omega_1$*.*

Proof. We want to show that if $\mu < \mu_B(\alpha)$, $\Omega(\mu,\omega_1;\alpha) = 0$ for all x, and if $\mu \geq \mu_B(\alpha)$ then $\Omega(\mu,\omega_1;\alpha) = \omega_1$.

Fix $\omega_1 > 0$. Suppose there exists $\mu_\omega(\alpha) < \mu_B(\alpha)$ such that $\Omega(\mu,\omega_1;\alpha) > 0 \iff \mu \geq \mu_\omega(\alpha)$. Arguments similar to those in Lemma 2 imply that $\Omega(\mu,\omega_1;\alpha) = \overline{\Omega}(\omega_1;\alpha)$ for all $\mu \geq \mu_\omega(\alpha)$. Let $\hat{\mu} \in (\mu_w(\alpha), \mu_B(\alpha))$ such that $\int_{\mu_w(\alpha)}^{\mu_B(\alpha)} \epsilon dG(\mu) = \int_{\hat{\mu}}^{\mu_B(\alpha)} \delta dG(\mu) \implies \delta = \frac{G(\mu_B(\alpha)) - G(\mu_w(\alpha))}{G(\mu_B(\alpha)) - G(\hat{\mu})}\epsilon$ We will take the limit $\epsilon \to 0$ and $\hat{\mu} \to \mu_B(\alpha)$ such that $\overline{T}_1(\alpha) > \delta > 0$ and finite. Consider the following alternative allocation:

$$\hat{\Omega}(\mu,\omega_1;\alpha) = \begin{cases} 0 & \text{if } \mu < \mu_w(\alpha) \\ \overline{\Omega}(\omega_1;\alpha) - \epsilon & \text{if } \mu \geq \mu_w(\alpha) \end{cases}$$

$$\hat{T}_1(\mu;\alpha) = \begin{cases} 0 & \text{if } \mu < \hat{\mu} \\ \delta & \text{if } \mu \in [\hat{\mu}, \mu_B(\alpha)) \\ \overline{T}_1(\alpha) & \text{if } \mu \geq \mu_B(\alpha) \end{cases}$$

and

$$\hat{c}_2(x_w;\alpha) = \begin{cases} c_2(x_w;\alpha) + [\mu_B(\alpha) - \mu_w(\alpha)]\epsilon + [\hat{\mu} - \mu_B(\alpha)]\delta & \text{if } \mu < \mu_w(\alpha) \\ c_2(x_w;\alpha) + [\hat{\mu} - \mu_B(\alpha)]\delta + \mu_B(\alpha)\epsilon & \text{if } \mu \in [\mu_w(\alpha), \hat{\mu}) \\ c_2(x_w;\alpha) - \mu_B(\alpha)(\delta - \epsilon) & \text{if } \mu \in [\hat{\mu}, \mu_B(\alpha)) \\ c_2(x_w;\alpha) & \text{if } \mu \geq \mu_B(\alpha) \end{cases}$$

where $x_w = (\mu, \omega_1, 0, 0)$. Similar arguments as in the the proof of Lemma 2 show that the alternative allocation satisfies the IC constraints. Finally, note that in the limit $\epsilon \to 0$ and $\hat{\mu} \to \mu_B(\alpha)$ with $\delta > 0$ and finite, we have

$$\hat{c}_2(x_w;\alpha) = \begin{cases} c_2(x_w;\alpha) & \text{if } \mu < \mu_w(\alpha) \\ c_2(x_w;\alpha) & \text{if } \mu \in [\mu_w(\alpha), \hat{\mu}) \\ c_2(x_w;\alpha) - \mu_B(\alpha)\delta & \text{if } \mu \in [\hat{\mu}, \mu_B(\alpha)) \\ c_2(x_w;\alpha) & \text{if } \mu \geq \mu_B(\alpha) \end{cases}$$

which implies that the planner can choose $\hat{T}_2(x_w;\alpha) < T_2(x_w;\alpha)$.

Now, suppose there exists $\mu_\omega(\alpha) > \mu_B(\alpha)$ such that $\Omega(\mu,\omega_1;\alpha) > 0 \iff \mu \geq \mu_\omega(\alpha)$. It should be clear that similar arguments as before imply that there exists an alternative allocation that increases the planner's objective and lowers the cost in period 2.

Finally, Assumption 3 implies that $\Omega(\mu,\omega_1;\alpha)$ is linear in ω_1 with a unit coefficient. Since $\Omega(\mu,0;\alpha) = 0$, this implies that $\Omega(\mu,\omega_1;\alpha) = \omega_1$. □

Lemma 18. *In the optimum*

$$\Xi(x;\alpha) = \begin{cases} 0 & \text{if } \mu < \mu_B(\alpha) \\ P_S(\alpha)[s_G(x;\alpha) + s_B(x;\alpha)] & \text{if } \mu \geq \mu_B(\alpha) \end{cases}$$

where $s_j(x;\alpha) = h_j$ *if* $\mu \geq \frac{Z_j}{P_S(\alpha)}$ *for* $j \in \{G, B\}$ *with* $Z_G = Z$ *and* $Z_B = \alpha Z$*.*

Proof. Truthful revelation of portfolios requires that $\Xi(x;\alpha)$ is weakly increasing in (h_G, h_B) and that there is an associated cost of getting transfers in period 1. Moreover, Assumption 3 and previous lemmas imply that $\Xi(x;\alpha) = 0$ for $\mu < \mu_B(\alpha)$. Otherwise, it would be optimal to either under- or over-report ω_1. Thus, the agents are rewarded only for the trees they give to the planner:

$$\Xi(x;\alpha) = \begin{cases} 0 & \text{if } \mu < \mu_B(\alpha) \\ \overline{P}_S(x;\alpha)[s_G(x;\alpha) + s_B(x;\alpha)] & \text{if } \mu \geq \mu_B(\alpha) \end{cases}$$

where we used that if $s_G > 0$ and $s_B > 0$, then the reward for trees has to be the same. Note that fixing (μ, ω_1), if s_j is increasing in h_j then $s_j = h_j$. Otherwise, agents will always have incentives to under- or over-report their portfolios. Then, Assumption 3 implies that when $s_j > 0$, $\overline{P}_S$ is independent of (h_G, h_B). Finally, Assumption 3 implies that $\Xi(x;\alpha)$ is independent of ω_1, hence $\overline{P}_S(x;\alpha) = P_S(\alpha)$. Finally, note that for $\mu \geq \mu_B$ it is optimal to truthfully report a portfolio with $h_j > 0$ when $s_j > 0$, if $\mu \geq \frac{Z_j}{P_S(\alpha)}$ where $Z_G = Z$ and $Z_B = \alpha Z$. □

Lemma 19. *Let $\tilde{\mu}_B(\alpha) \equiv \max\left\{\mu_B(\alpha), \frac{\alpha Z}{P_S(\alpha)}\right\}$. If $\tilde{\mu}_B(\alpha) = \mu_B(\alpha)$, consumption in period 2 is*

$$c_2(x;\alpha) = \begin{cases} W_2 + \mu_B\left[\omega_1 + \overline{T}_1(\alpha) + P_S(\alpha) h_B\right] + Zh_G - \mathbb{T}_2(\alpha) & \text{if } \mu \le \mu_B(\alpha) \\ W_2 + Zh_G - \mathbb{T}_2(\alpha) & \text{if } \mu \in \left[\tilde{\mu}_B(\alpha), \frac{Z}{P_S(\alpha)}\right) \\ W_2 - \mathbb{T}_2(\alpha) & \text{if } \mu \ge \frac{Z}{P_S(\alpha)} \end{cases}$$

If instead $\tilde{\mu}_B(\alpha) = \frac{\alpha Z}{P_S(\alpha)}$, consumption in period 2 is

$$c_2(x;\alpha) = \begin{cases} W_2 + \mu_B\left[\omega_1 + \overline{T}_1(\alpha)\right] + Zh_G - \mathbb{T}_2(\alpha) & \text{if } \mu \le \mu_B(\alpha) \\ W_2 + Zh_G + \alpha Z h_B - \mathbb{T}_2(\alpha) & \text{if } \mu \in [\mu_B(\alpha), \tilde{\mu}_B(\alpha)) \\ W_2 + Zh_G - \mathbb{T}_2(\alpha) & \text{if } \mu \in \left[\tilde{\mu}_B(\alpha), \frac{Z}{P_S(\alpha)}\right) \\ W_2 - \mathbb{T}_2(\alpha) & \text{if } \mu \ge \frac{Z}{P_S(\alpha)} \end{cases}$$

Proof. The allocation of consumption in period 2 is irrelevant for the planner except for two issues: satisfy the IC constraints and minimize $\mathbb{T}_2(\alpha)$. When $\mu \ge \frac{Z}{P_S(\alpha)}$, $c_1(x;\alpha) = \omega_1 + \overline{T}_1(\alpha) + P_S(\alpha)[h_G + h_B]$ is the highest so $c_2(x;\alpha)$ is the lowest. Reducing $c_2(x;\alpha)$ in this case relaxes the IC constraint, so $c_2(x;\alpha) = W_2 - \mathbb{T}_2(\alpha)$. Consider now $\mu \in \left[\tilde{\mu}_B(\alpha), \frac{Z}{P_S(\alpha)}\right)$. Then, $c_1(x;\alpha) = \omega_1 + \overline{T}_1(\alpha) + P_S(\alpha) h_B$. Then, IC requires

$$\lim_{\mu \to \frac{Z}{P_S(\alpha)}^-} \mu\left[\omega_1 + \overline{T}_1(\alpha) + P_S(\alpha) h_B\right] + c_2(x;\alpha) = \lim_{\mu \to \frac{Z}{P_S(\alpha)}^+} \mu\left[\omega_1 + \overline{T}_1(\alpha) + P_S(\alpha)[h_G + h_B]\right] + W_2 - \mathbb{T}_2(\alpha)$$

or, after some algebra,

$$\lim_{\mu \to \frac{Z}{P_S(\alpha)}^-} c_2(x;\alpha) = W_2 + Zh_G - \mathbb{T}_2(\alpha)$$

Since all agents in the interval consume the same in period 1, $c_2(x;\alpha) = W_2 + Zh_G - \mathbb{T}_2(\alpha)$ for all $\mu \in \left[\tilde{\mu}_B(\alpha), \frac{Z}{P_S(\alpha)}\right)$. Consider now $\mu \in [\mu_B(\alpha), \tilde{\mu}_B(\alpha))$. There are two cases. If $\tilde{\mu}_B(\alpha) = \mu_B(\alpha)$, the interval is empty. If $\tilde{\mu}_B\left(\frac{\alpha Z}{P_S(\alpha)}\right)$ the interval is non-empty. Consider the latter case. Then

$$\lim_{\mu \to \frac{\alpha Z}{P_S(\alpha)}^-} \mu\left[\omega_1 + \overline{T}_1(\alpha)\right] + c_2(x;\alpha) = \lim_{\mu \to \frac{\alpha Z}{P_S(\alpha)}^+} \mu\left[\omega_1 + \overline{T}_1(\alpha) + P_S(\alpha) h_B\right] + W_2 + Zh_G - \mathbb{T}_2(\alpha)$$

or, after some algebra,

$$\lim_{\mu \to \frac{\alpha Z}{P_S(\alpha)}^-} c_2(x;\alpha) = W_2 + Zh_G + \alpha Z h_B - \mathbb{T}_2(\alpha)$$

Since all agents in the interval consume the same in period 1, $c_2(x;\alpha) = W_2 + Zh_G + \alpha Z h_B - \mathbb{T}_2(\alpha)$ for all $\mu \in [\mu_B(\alpha), \tilde{\mu}_B(\alpha))$. Finally, consider $\mu \le \mu_B(\alpha)$. Then $c_1(x;\alpha) = 0$. Again, there are two cases depending on the value of $\tilde{\mu}_B(\alpha)$. Suppose $\tilde{\mu}_B(\alpha) = \frac{\alpha Z}{P_S(\alpha)}$. Then, IC requires

$$\lim_{\mu \to \mu_B^-} c_2(x;\alpha) = \lim_{\mu \to \mu_B(\alpha)^+} \mu\left[\omega_1 + \overline{T}_1(\alpha)\right] + W_2 + Zh_G + \alpha Z h_B - \mathbb{T}_2(\alpha)$$

hence

$$\lim_{\mu \to \mu_B^-} c_2(x;\alpha) = W_2 + \mu_B(\alpha)\left[\omega_1 + \overline{T}_1(\alpha)\right] + Zh_G + \alpha Z h_B - \mathbb{T}_2(\alpha)$$

If $\tilde{\mu}_B(\alpha) = \mu_B(\alpha)$, then IC requires

$$\lim_{\mu \to \mu_B^-} c_2(x;\alpha) = \lim_{\mu \to \mu_B(\alpha)^+} \mu\left[\omega_1 + \overline{T}_1(\alpha) + P_S(\alpha) h_B\right] + W_2 + Zh_G + \alpha Z h_B - \mathbb{T}_2(\alpha)$$

hence

$$\lim_{\mu \to \mu_B^-} c_2(x;\alpha) = W_2 + \mu_B(\alpha)\left[\omega_1 + \overline{T}_1(\alpha) + P_S(\alpha) h_B\right] + Zh_G - \mathbb{T}_2(\alpha)$$

□

Lemma 20. *In the optimum,*

$$\mu_B(\alpha)\left[\omega_1 + \overline{T}_1(\alpha) + P_S(\alpha) h_B\right] = \lambda_M(\alpha) m(\alpha) Z + [1 - \lambda_M(\alpha)] m(x;\alpha) \alpha Z + \mathcal{T}_2(x;\alpha)$$

if $\tilde{\mu}_B(\alpha) = \mu_B(\alpha)$, *and*

$$\mu_B(\alpha)\left[\omega_1 + \overline{T}_1(\alpha)\right] = \lambda_M(\alpha)\, m(x;\alpha)\, Z + \left[1-\lambda_M(\alpha)\right] m(x;\alpha)\,\alpha Z + \mathcal{T}_2(x;\alpha)$$

if $\tilde{\mu}_B(\alpha) = \frac{\alpha Z}{P_S(\alpha)}$, *where*

$$m(x;\alpha) = \int_x \left[s_G(x;\alpha) + s_B(x;\alpha)\right] d\Gamma(x)$$

and

$$\lambda_M(x;\alpha) = \frac{\int_x s_G(x;\alpha)\, d\Gamma(x)}{\int_x \left[s_G(x;\alpha) + s_B(x;\alpha)\right] d\Gamma(x)}.$$

Proof. Immediate from the previous analysis. The planner has to compensate the agents with $\mu < \mu_B(\alpha)$ for not consuming in period 1. It has to ways of doing this: transfers $\mathcal{T}_2(x;\alpha)$ and trees, $\int_x s_G(x;\alpha)\, d\Gamma(x)$ and $\int_x s_B(x;\alpha)\, d\Gamma(x)$. Without loss of generality, we can assume that it distributes tree quality proportionally to how much it collected, $\lambda_M(x;\alpha)$. □

Lemma 21. *The resource constraint in* $t = 1$ *implies*

$$G(\mu_B(\alpha))\, W_1 = \left[1 - G(\tilde{\mu}_B(\alpha))\right] H_B P_S(\alpha) + \left[1 - G\left(\frac{Z}{P_S(\alpha)}\right)\right] H_G P_S(\alpha) + \left[1 - G(\mu_B(\alpha))\right] \overline{T}_1(\alpha)$$

Proof. We have

$$\int_x c_1(x;\alpha)\, d\Gamma(x) = \int_{\mu_B(\alpha)}^{\tilde{\mu}_B(\alpha)} \left[W_1 + \overline{T}_1(\alpha)\right] dG(\mu) + \int_{\tilde{\mu}_B(\alpha)}^{\frac{Z}{P_S(\alpha)}} \left[W_1 + \overline{T}_1(\alpha) + P_S(\alpha) H_B\right] dG(\mu) + \int_{\frac{Z}{P_S(\alpha)}}^{\mu^{\max}} \left[W_1 + \overline{T}_1(\alpha) + P_S(\alpha)\left[H_G + H_B\right]\right] dG(\mu)$$

Since the resource constraint is $\int_x c_1(x;\alpha)\, d\Gamma(x) = W_1$, replacing and after doing some algebra we get the desired result. □

Lemma 22. *In the optimum,*

$$\mathbb{T}_2(\alpha) = (1+\chi)\, \mu_B(\alpha)\left[\left[P_S(\alpha) - P_B(\alpha)\right] S(\alpha) + \overline{T}_1(\alpha)\right].$$

Proof. Suppose $\tilde{\mu}_B(\alpha) = \mu_B(\alpha)$. Resource constraint in period 2 implies

$$G(\mu_B(\alpha))\, \mu_B(\alpha)\left[W_1 + \overline{T}_1(\alpha) + P_S(\alpha) H_B\right] = \mu_B(\alpha)\, P_B(\alpha)\, S(\alpha) + \int_x \mathcal{T}_2(x;\alpha)\, d\Gamma(x)$$

where $P_B(\alpha) \equiv \frac{\lambda_M(\alpha) Z + [1-\lambda_M(\alpha)]\alpha Z}{\mu_B(\alpha)}$ and $S(\alpha) = \left[1 - G\left(\frac{Z}{P_S(\alpha)}\right)\right] H_G + H_B$. Recall that the resource constraint in period 1 implies

$$W_1 = \frac{1}{G(\mu_B(\alpha))}\left[\left[1 - G(\mu_B(\alpha))\right] H_B P_S(\alpha) + \left[1 - G\left(\frac{Z}{P_S(\alpha)}\right)\right] H_G P_S(\alpha) + \left[1 - G(\mu_B(\alpha))\right] \overline{T}_1(\alpha)\right]$$

Replacing into the resource constraint in period 2 and after some algebra we get the desired result. If instead $\tilde{\mu}_B(\alpha) = \frac{\alpha Z}{P_S(\alpha)}$, the resource constraint in period 2 implies

$$G(\mu_B(\alpha))\, \mu_B(\alpha)\left[W_1 + \overline{T}_1(\alpha)\right] = \mu_B(\alpha)\, P_B(\alpha)\, S(\alpha) + \int_x \mathcal{T}_2(x;\alpha)\, d\Gamma(x)$$

The resource constraint in period 1 implies

$$W_1 = \frac{1}{G(\mu_B(\alpha))}\left[\left[1 - G(\tilde{\mu}_B(\alpha))\right] H_B P_S(\alpha) + \left[1 - G\left(\frac{Z}{P_S(\alpha)}\right)\right] H_G P_S(\alpha) + \left[1 - G(\mu_B(\alpha))\right] \overline{T}_1(\alpha)\right]$$

Replacing into the resource constraint in period 2 and after some algebra we get the desired result. □

Lemma 23. *Let* $\omega(\alpha) = \frac{P_S(\alpha)}{P_B(\alpha)}$ *and* $B(\alpha) = \mu_B(\alpha)\, \overline{T}_1(\alpha)$. *Then the planner's problem is equivalent to Ramsey problem with* $\omega(\alpha)$ *and* $B(\alpha)$ *as instruments.*

Proof. Immediate from previous lemmas and noting that the shadow values coincide. $\square$

$\square$

B Planner's problem under different degrees of information incompleteness

I first find the solution to a relaxed version in which the planner can observe agents' μ but not the agents' portfolios, trades or production decisions. It turns out that by observing μ, the planner can implement the (*ex-ante*) first best allocation. Then, I show that first best is not feasible when μ is unobservable. These exercises clarify the role of the different information frictions in this economy.

Observable μ

Consider the program (PP′), but assume that the planner can observe agents' type μ, so that the optimal plan does not need to satisfy the IC constraints. The tree quality production and the agents' portfolios are still their private information, and trade of trees takes place in an anonymous market. Under these assumptions, the planner can implement the *ex-ante* first best allocation.

Lemma 24. *Consider a planner that can observe the agents' individual type μ, but cannot observe agents' tree quality production or their portfolios, and trade of trees takes place in an anonymous market. The solution to the planner's problem is a solution to the first best program* (FB)*. The volume traded in the market for trees is zero.*

Proof. Consider the following plan:

$$P_S(\alpha) = 0, P_B(\alpha) > Z$$

$$T_1(\mu, \alpha) = \begin{cases} -W_1 & \text{if } \mu < \tilde{\mu} \\ \frac{G(\tilde{\mu})W_1}{1-G(\tilde{\mu})} & \text{if } \mu \geq \tilde{\mu} \end{cases}$$

$$T_2(\mu, \alpha) = 0$$

for some $\tilde{\mu} > 1$. With these prices and transfers, the volume traded in the market for trees is zero. Thus, $\gamma_G = Z$ and $\gamma_B = E[\alpha] Z$. By Assumption 1, this implies $H_B = 0$. Moreover, the transfers are feasible since

$$\int_1^{\mu^{\max}} T_1(\mu, \alpha) dG(\mu) = -G(\tilde{\mu}) W_1 + [1 - G(\tilde{\mu})] \frac{G(\tilde{\mu}) W_1}{1 - G(\tilde{\mu})} = 0$$

Setting $\tilde{\mu} = \mu^{\max}$, the plan implements the first best.

Suppose there is a plan that implements the first best with positive volume traded. For trade to happen, it must be that $P_S(\alpha) \geq \frac{Z}{\mu^{\max}}$, as only good trees are produced in the first best allocation. Since first best implies that $\mu_B(\alpha) = \mu^{\max}$, we get

$$\gamma_B \geq \gamma_G \geq Z$$

which contradicts that only good trees are produced. Thus, the first best implementation requires to shut down the market for trees. □

The economy features two different sources of information asymmetry. First, agents have private information about the quality of the trees they produce in period 0 and they trade in period 1. Second, they have private information about their liquidity needs in period 1, given by μ. These two frictions, combined with the anonymity in the market for trees, leads to a *laissez-faire* equilibrium that features inefficient production of trees, and the exposure of the economy to abrupt financial crises. Lemma 24 shows, however, that by observing agents' type μ, the planner can achieve a first best allocation, even in the presence of the other sources of information asymmetry. The optimal plan is as follows: the planner sets $T_1(\mu, \alpha) = -W_1$ for all $\mu < \mu^{\max}$ and transfers all the resources to the agents with $\mu = \mu^{\max}$. This way, the allocation of consumption is optimal and the demand for trees is zero, effectively shutting down the private market, and making the information asymmetry about tree quality inconsequential. Since bad trees are produced only to be sold in the secondary market, the induced equilibrium features only production of good trees. That is, the optimal policy separates the liquidity value of assets from their dividend value, so that trees are produced only for fundamental reasons. Crucial to this solution is the planner's "taxation" power (through transfers), which allows it to bypass the agents' limited commitment problem.[39]

In contrast, when μ is unobservable, the resource requirements of the IC constraints makes the first best allocation infeasible.

[39] Note, however, that this policy does not implement an *interim* Pareto improvement, since agents with $\mu < \mu^{\max}$ are worse off after the planner's intervention.

Unobservable μ

Now, assume that the planner cannot observe μ. Even if the deadweight loss from transfers is zero, i.e., $\chi = 0$, and the production of trees is observable, the planner cannot achieve a first best allocation.

Lemma 25. *Consider the problem of a planner that cannot observe the agents' individual type μ, but it can observe the agents' tree quality production and portfolios. Suppose that the deadweight loss from transfers is zero (i.e., $\chi = 0$). The solution to the planner's problem is not a solution of the first best program* (FB).

Proof. First, suppose that the market for trees is inactive. From incentive compatibility, the transfers in period 1 that implement the first best require the following transfers in period 2:

$$\mathcal{T}_2(\mu, \alpha) = \tilde{\mu}\left[T_1(\tilde{\mu}, \alpha) + W_1\right] \quad \forall\, \mu < \tilde{\mu}$$

Therefore

$$\mathbb{T}_2(\alpha) = -(1+\chi)G(\tilde{\mu})\tilde{\mu}\left[T_1(\tilde{\mu}, \alpha) + W_1\right] = -(1+\chi)\frac{G(\tilde{\mu})\tilde{\mu}}{1 - G(\tilde{\mu})}W_1.$$

But

$$\lim_{\tilde{\mu} \to \mu^{\max}} \mathbb{T}_2(\alpha) = -\infty$$

So the transfers are not feasible. It is immediate to see that any other transfer schedule different than the one in Lemma 24 is either infeasible or does not reallocate all the endowment to the agents with $\mu = \mu^{\max}$.

Now, suppose the market for trees is active. Since the planner can observe the production of trees, it can implement $\lambda_E = 1$. Moreover, a positive demand for trees in state α requires $T_1(\mu, \alpha) > -W_1$ for some $\mu < \mu^{\max}$. In that case, market clearing implies

$$\int_1^{\mu_B(\alpha)} \left[W_1 + T_1(\mu, \alpha)\right] dG(\mu) = \left[1 - G(\mu_B(\alpha))\right] H_G P_B(\alpha)$$

Given that $\lambda_M(\alpha) = 1$, $P_B(\alpha) = \frac{Z}{\mu_B(\alpha)}$, and we get

$$\frac{1 - G(\mu_B)}{G(\mu_B)\mu_B} = \frac{W_1 + \frac{\int_1^{\mu_B} T_1(\mu,\alpha) dG(\mu)}{G(\mu_B)}}{Z H_G} > 0$$

since $\frac{\int_1^{\mu_B} T_1(\mu,\alpha) dG(\mu)}{G(\mu_B)} > -W_1$ by construction. Hence, $\mu_B < \mu^{\max}$ and the first best is not implemented. □

Recall that first best requires that the planner transfer all the endowment to the agents with $\mu = \mu^{\max}$. When the planner cannot observe agents' type μ, any transfer intended for one type has to be made available to all other types. Because of the assumption that agent-specific transfers in period 2 have to be non-negative, the IC constraints imply that all agents with $\mu < \mu^{\max}$ have to be promised at least $\mu^{\max}[T_1(\mu^{\max}, \alpha) + W_1]$ in period 2, which is what the marginal agent is giving up by not choosing the transfers intended for $\mu = \mu^{\max}$. It turns out that this promise requires an infinite amount of resources. This argument resembles Bewley (1983), who finds that in a model with idiosyncratic risk and infinitely lived agents, the quantity of "money" (which pays a positive interest, so it resembles government bonds) necessary for agents to self-insure is infinite, so first best is not feasible. I extend Bewley's result to a more general transfer scheme and finitely lived agents. Moreover, the result does not rely on a positive deadweight loss of transfers.

C The Ramsey Problem

Consider the economy described in Section 2, where the government has access to two sets of tools. First, it can issue state-contingent bonds in period 0, which mature in period 2. Crucially, these bonds can be traded in (frictionless) secondary markets in period 1. Second, the government can introduce a transaction subsidy or tax (more generally, a *wedge*) in the market for trees. Note that the government can choose to shut down the private market by setting a sufficiently high tax. Thus, the presence of the market for trees does not imply an additional restriction in the government's problem. Naturally, the government needs to satisfy a budget constraint in each period. I assume that only lump-sum taxes in period 2 are available to the government, which entail a proportional deadweight loss.

Formally, let a Ramsey plan $\mathcal{P}^R = (\{B(\alpha), \omega(\alpha)\}_{\forall\alpha})$ be a set of state-contingent government bonds issued in period 0 and that mature in period 2, where $B(\alpha)$ denotes the bond's face value in the aggregate state α, and a set of *market wedge*, $\omega(\alpha)$, which induce the prices $P_S(\alpha)$ and $P_B(\alpha)$ in the market for trees, where $P_S(\alpha)$ denotes the price received by sellers and $P_B(\alpha)$ denotes the price paid by buyers, and $P_S(\alpha) = (1+\omega(\alpha))P_B(\alpha)$. For a given choice of Ramsey plan $\mathcal{P}^R$, the characterization of the economy's equilibrium is analogous to the one in *laissez-faire* studied in the previous section. Next, I describe the set of constraints that the economy imposes on the government's problem.

Agents' Problem. The agents' value function in period 2 is analogous to that in Section 2, except that government bonds are state-contingent now, that is,

$$V_2(h_G, h_B, b(\alpha); X) = W_2 + Zh_G + \alpha Z h_B + b(\alpha) - \mathbb{T}_2(\alpha).$$

Let's turn to period 1. Now, we need to distinguish between the price paid by buyers and the price received by sellers. The agents' problem is state X is given by

$$V_1(h_G, h_B, b(\alpha); \mu, X) = \max_{c,m,s_G,s_B,h'_G,h'_B,b'(\alpha)} \mu c + V_2(h'_G, h'_B, b'(\alpha); X),$$

subject to

$$\begin{aligned} c + P_B(X)m + Q_1^B(X)(b'(\alpha) - b(\alpha)) &\leq W_1 + P_S(X)(s_G + s_B), \\ h'_G &= h_G + \lambda_M(X)m - s_G, \\ h'_B &= h_B + (1-\lambda_M(X))m - s_B, \\ c \geq 0, \quad m \geq 0, \quad b'(\alpha) \geq 0, &\quad s_G \in [0, h_G], \quad s_B \in [0, h_B]. \end{aligned}$$

The solution to this problem is now characterized by three thresholds:

i. Agents consume if $\mu \geq \mu_B(X)$, where

$$\mu_B(X) = r_M(X) = \frac{\lambda_M(X)Z + (1-\lambda_M(X))\alpha Z}{P_B(X)} = \frac{1}{Q_1^B(X)}$$

ii. All agents sell their good trees if $P_S(X) \geq \frac{\alpha Z}{\mu_B(X)}$. If $P_S(X) < \frac{\alpha Z}{\mu_B(X)}$, only agents with $\mu \geq \frac{\alpha Z}{P_S(X)}$ sell their bad trees.

iii. Agents sell their good trees only if $\mu \geq \frac{Z}{P_S(X)}$.

Relative to the model in Section 2, now we might have that not all agents sell their bad trees, as the adverse selection subsidy can be negative if the transaction tax is sufficiently high.

Finally, the agents' problem in period 0 is

$$V_0 = \max_{h_G, h_B, \{b(\alpha)\}_{\forall\alpha}} E[V_1(h_G, h_B, b(\alpha); \mu, X)],$$

subject to

$$\begin{aligned} h_B + C(h_G) + \int_{\underline{\alpha}}^{\overline{\alpha}} Q_0^B(\alpha) b(\alpha) dF(\alpha) &\leq W_0 + T_0, \\ h_G \geq 0, \quad h_B \geq 0, &\quad b(\alpha) \geq 0. \end{aligned}$$

Let

$$\begin{aligned} \gamma_G &\equiv E[\max\{\mu P_S(X), Z\}] \\ \gamma_B &\equiv E[\max\{\mu, \tilde{\mu}_B(X)\} P_S(X)], \end{aligned}$$

where $\tilde{\mu}_B(X) \equiv \max\left\{\mu_B(X), \frac{\alpha Z}{P_S(X)}\right\}$. Then, the solution to the agents' problem in period 0 is given by

$$\frac{\gamma_G}{\gamma_B} = C'(h_G) h_B = W_0 - C(h_G).$$

Planner's Budget Constraints. The planner's budget constraint in period 0 is

$$\int_{\underline{\alpha}}^{\overline{\alpha}} Q_0^B(\alpha) B(\alpha) dF(\alpha) = T_0.$$

The budget constraint in period 1 is

$$[P_S(X) - P_B(X)]S(X) = Q_1^B(\alpha)[B'(\alpha) - B(\alpha)]$$

Finally, the budget constraint in period 2 is

$$\mathbb{T}_2(X) = (1+\chi)B'(\alpha) = (1+\chi)\left[B(\alpha) + \frac{P_S(X) - P_B(X)}{Q_1^B(\alpha)} S(X)\right].$$

Market for Trees. Given a plan $\mathcal{P}^R$, an active market equilibrium must satisfy

$$\mu_B(X) = \frac{\lambda_M(X)Z + (1-\lambda_M(X))\alpha Z}{P_B(X)}, \tag{18}$$

$$\lambda_M(X) = \frac{[1-G(\mu_S(X))]H_G}{S(X)} \tag{19}$$

and

$$D(X) = S(X) \tag{20}$$

where

$$D(X) = \begin{cases} \frac{G(\mu_B(X))\left[W_1 + P_S(X)H_B + Q_1^B(\alpha)B(\alpha)\right] - Q_1^B(\alpha)B'(\alpha)}{P_B(X)} & \text{if } P_S(X) \geq \frac{\alpha Z}{\mu_B(X)} \\ \frac{G(\mu_B(X))\left[W_1 + Q_1^B(\alpha)B(\alpha)\right] - Q_1^B(\alpha)B'(\alpha)}{P_B(X)} & \text{if } P_S(X) < \frac{\alpha Z}{\mu_B(X)} \end{cases} \tag{21}$$

$$S(X) = \begin{cases} [1-G(\mu_S(X))]H_G + H_B & \text{if } P_S(X) \geq \frac{\alpha Z}{\mu_B(X)} \\ [1-G(\mu_S(X))]H_G + \left[1 - G\left(\frac{\alpha Z}{P_S(X)}\right)\right] H_B & \text{if } P_S(X) < \frac{\alpha Z}{\mu_B(X)} \end{cases}. \tag{22}$$

Note that using the planner's resource constraint in period 1, we can write the market clearing condition in the market for trees (equations (20)-(22)) as

$$G(\mu_B(X))W_1 = [1 - G(\tilde{\mu}_B(X))]\, H_B P_S(X) + [1 - G(\mu_S(X))]H_G P_S(X) + [1 - G(\mu_B(X))]\, \frac{B(\alpha)}{\mu_B(X)}$$

where I used that, in equilibrium, $Q_1^B(\alpha) = \frac{1}{\mu_B(X)}$.

D The Role of χ

The objective of this appendix is twofold. First, I show that assuming $\chi > 0$ and a non-binding constraint $\mathbb{T}_2(\alpha) \leq W_2$ is equivalent to assuming $\chi = 0$ and an appropriately chosen state-contingent endowment in period 2, $W_2(\alpha)$. Second, I discuss an alternative interpretation of χ. Agents are allowed to borrow in period 1, but there is a cost χ of enforcing contracts, both for the private sector and for the government.

D.1 The case of $\chi = 0$

Consider the Ramsey problem with $\chi = 0$ and assume that the endowment in period 2 is state-contingent, that is, we have $W_2(\alpha)$. Note that constant W_2 is a special case of this general formulation. Naturally, we cannot restrict attention to solutions where the constraint $\mathbb{T}_2(\alpha) \leq W_2(\alpha)$ is not binding. Let $\zeta(\alpha)$ denote the Lagrange multiplier associated with this constraint. Then, we can rewrite the problem as follows:

$$\max_{\{B(\alpha),\omega(\alpha)\}_{\forall\alpha}, H_G} E\left[\boldsymbol{\mathcal{U}_1}(\alpha) + ZH_G + \alpha Z(W_0 - C(H_G)) - \zeta(\alpha)\left(\mathbb{T}_2(\alpha) - W_2(\alpha)\right)\right]$$

subject to

$$C'(H_G) = \frac{\gamma_G}{\gamma_B}.$$

It is immediate to see that this problem is equivalent to one in which χ is state-contingent, and $W_2(\alpha)$ is chosen so that $W_2(\alpha) = \mathbb{T}_2(\alpha)$ for all α. In particular, the trade-offs for the planner remain the same. Thus, we can rethink the exercise in the main text as assuming $W_2(\alpha)$ such that $\zeta(\alpha) = \frac{\chi}{1+\chi}$ and $W_2(\alpha) = \mathbb{T}_2(\alpha)$. However, the analysis in Appendix B ("Unobservable μ") shows that as $W_2 \to \infty$, the planner can get arbitrarily close to the first best allocation. In that case, the planner can provide unlimited public liquidity in period 1 so that only the agents with the highest μ consume in period 1, while all other agents consume in period 2. In this case, the liquidity premium drops to zero and no agents sell their good trees, which effectively breaks down the market for all realizations of α. Assumption 1 then guarantees that only good trees are produced.

D.2 The cost of enforcing private contracts

Consider an economy like the one in Section 2, but where agents can trade private bonds with each other. Private bonds are risk free and trees cannot be collateralized. Moreover, there is a cost of enforcing contracts. Let D be the face value of a private bond. Then, the lender receives $\frac{D}{1+\chi}$ in period 2, with $\chi > 0$.[40] Let $Q_1^D(X)$ denote the price of a unit of private bond in period 1, state X. Moreover, I assume that government bonds are state-contingent. The problem of the agents in period 1 is

$$V_1(h_G, h_B, b(\alpha); \mu, X) = \max_{\substack{c_1, c_2, m, s_G, s_B, \\ h'_G, h'_B, b'(\alpha), d}} \mu c_1 + c_2$$

subject to

$$c_1 + P_M(X)(m - s_G - s_B) + Q_1^B(X)(b'(\alpha) - b(\alpha)) \leq W_1 + Q_1^D(X)\, d,$$
$$c_2 \leq W_2 + Zh'_G + \alpha Z h'_B + b'(\alpha) - \max\{d, 0\} + \frac{\min\{d, 0\}}{1+\chi} - \mathbb{T}_2(X)$$
$$h'_G = h_G + \lambda_M(X)m - s_G,$$
$$h'_B = h_B + (1 - \lambda_M(X))m - s_B,$$
$$c_t \geq 0, \quad m \geq 0, \quad b'(\alpha) \geq 0, \quad s_G \in [0, h_G], \quad s_B \in [0, h_B], \quad d \leq W_2 - \mathbb{T}_2(X)$$

The first order condition with respect to d is given by

$$Q_D(X) = \frac{1+\eta}{\mu} \quad \text{if } d \geq 0$$

and

$$Q_D(X) = \frac{1}{1+\chi}\frac{1}{\mu_B(X)} \quad \text{if } d < 0.$$

Let $\mu_P(X) \equiv (1+\chi)\mu_B(X)$. Then, for $\mu \geq \mu_P(X)$, $d = W_2$, and for $\mu \in [\mu_B(X), \mu_P(X))$, $d = 0$. That is, not all agents that consume in period 1 borrow; only those with a sufficiently high μ do it, as the interest rate on private debt includes an adjustment for the cost of enforcing the contracts. Note, however, that this is not the case for government debt. Even though the government also needs to pay χ per unit of bond, this cost does not appear as a wedge in the secondary market for government bonds. Thus, all agents with $\mu \geq \mu_B(X)$ will sell their government bonds to consume. The budget constraint of the government in period 2 is

$$\mathbb{T}_2(X) = (1+\chi)B(\alpha).$$

Solving the model following the same steps as in Section 3, we can obtain the following *ex-ante* welfare function for the agents as a function of the government debt, $\{B(\alpha)\}_{\forall\alpha}$:

$$\mathbb{W}(\{B(\alpha)\}_{\forall\alpha}) = E\left[\int_{\mu_B(X)}^{\mu^{\max}} \mu \left[W_1 + P_M(X) H_B + \frac{B(\alpha)}{\mu_B(X)}\right] dG(\mu) + \int_{\mu_P(X)}^{\mu^{\max}} \mu \frac{W_2 - (1+\chi)B(\alpha)}{\mu_P(X)} dG(\mu) + \right.$$
$$\left. \int_{\mu_S(X)}^{\mu^{\max}} \mu P_M(X) H_G dG(\mu) + \frac{1+\chi G(\mu_P(X))}{1+\chi} W_2 - \chi G(\mu_P(X)) B(\alpha) + ZH_G + \alpha Z H_B \right]$$

Suppose $\chi = 0$. Then, it is immediate to see that government bonds and private debt are equivalent. In particular, the welfare function becomes

$$\mathbb{W}(\{B(\alpha)\}_{\forall\alpha}) = E\left[\int_{\mu_B(X)}^{\mu^{\max}} \mu \left[W_1 + P_M(X) H_B + \frac{W_2}{\mu_B(X)}\right] dG(\mu) + \int_{\mu_S(X)}^{\mu^{\max}} \mu P_M(X) H_G dG(\mu) + \right.$$
$$\left. W_2 + ZH_G + \alpha Z H_B \right]$$

which is independent of $\{B(\alpha)\}_{\forall\alpha}$. Does this mean that private debt and government bonds are equivalent? Exploring the case with $\chi > 0$ makes it clear that they are not. There are three main differences. First, the presence of a wedge. Because of the enforcement cost, the rate of return paid by borrowers is higher than the net return for lenders. This difference effectively reduces the set of agents that find it optimal to borrow, as $\mu_P(X) > \mu_B(X)$. In contrast, the

[40] Note that this implies a cost $\frac{\chi}{1+\chi}$ per unit of bond.

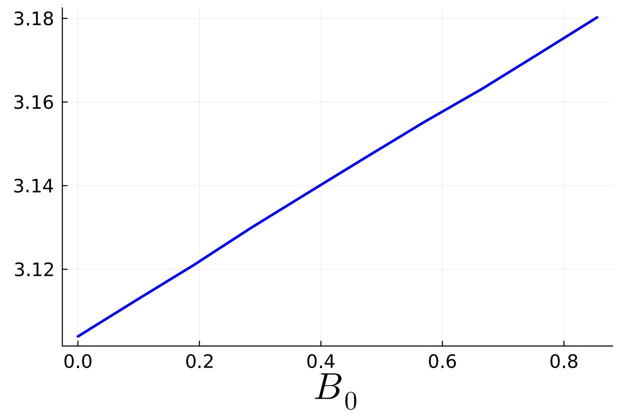

Figure 7: Agents' expected utility in period 0 as a function of B_0

Note: I assume $\alpha \sim U[0,1]$ and $\mu \sim U[1, \mu^{\max}]$ with $\mu^{\max} = 2$. Moreover, $C(H_G) = \xi \frac{(1+H_G)^{1+\nu}-1}{1+\nu}$, with $\xi = 0.2$ and $\nu = 6$. The other parameters of the model are: $W_0 = 0.25$, $W_1 = W_2 = 1$, and $Z = 1.3$. The cost of enforcing contracts is $\chi = 0.17$.

enforcement cost for government debt is not priced in the interest rate. The government needs to pay χ per unit of bond that it issues, but it does not introduce a wedge in the secondary market (it would be suboptimal to do so). Thus, government debt allows agents with $\mu \in [\mu_B(X), \mu_P(X))$ to increase their consumption. The second difference concerns the deadweight loss of the different types of bonds. Note that private debt is only issued by relatively high-μ agents. Hence, the total cost of enforcing private debt is given by

$$\frac{\chi}{1+\chi}\left[1 - G\left(\mu_P\left(X\right)\right)\right] d,$$

where d is the amount borrowed. In contrast, government bonds are issued in period 0, so all agents buy them, independently of whether they will end up needing them. The cost of issuing bonds is

$$\chi B(\alpha).$$

To see that government bonds are more costly, consider the following two extreme cases. First, suppose that government bonds are zero. Then, $d = W_2$, the agents that borrow receive $\frac{W_2}{(1+\chi)\mu_B(X)}$ in period 1, and the total cost of enforcement is

$$\frac{\chi}{1+\chi}\left[1 - G\left(\mu_P\left(X\right)\right)\right] W_2,$$

Now, suppose government bonds are maximized, so that $B(\alpha) = \frac{W_2}{1+\chi}$. Then, the agents that sell their bonds receive $\frac{W_2}{(1+\chi)\mu_B(X)}$ (the same as with private debt), and the cost of enforcement is

$$\frac{\chi}{1+\chi} W_2 > \frac{\chi}{1+\chi}\left[1 - G\left(\mu_P\left(X\right)\right)\right] W_2.$$

That is, because government bonds are *untargeted*, they are more costly.

The third difference between private debt and government bonds is the effects internalized in the decision to issue one or the other. For private debt, agents compare the liquidity benefits of taking debt in period 1 versus the cost in terms of forgone consumption in period 2. That is, the decision is purely in terms of *liquidity motives*. In contrast, a benevolent planner internalizes that a higher supply of government bonds reduces the liquidity premium in period 1, reducing the incentives to produce bad trees in period 0. Thus, the planner will choose to issue more government debt that the level prescribed by liquidity considerations only. This logic was present in Section 4, specifically in Proposition 6, where we found that the liquidity effect for government bonds is negative in the optimum.

Figure 7 presents a numerical example. The objective is not to compute the optimal policy, but to show that issuing government bonds can be welfare enhancing. In particular, I consider the effects of the following policy:

$$B(\alpha) = \begin{cases} B_0 > 0 & \text{if } \alpha < \overline{\alpha} \\ 0 & \text{otherwise} \end{cases}$$

for different levels of B_0. I then compute the welfare level according to $\mathbb{W}(\cdot)$. The figure shows that, for this example, welfare is increasing in B_0.

E Ex-ante interventions

The analysis in Section 4 assumes that the planner cannot observe the agents' portfolios in any period. This assumption is crucial for period 1, as the analysis requires that agents have the option to hide their good trees. However, the role of this assumption is less clear for period 0. It is natural to assume that the planner cannot distinguish the quality of the trees being produced. But if the *total* production of trees were observable, the planner could choose to limit it to improve welfare. In this appendix, I show that in the economy of Section 2, the planner never finds it optimal to restrict the production of trees, even though in the current model, the restriction would exclusively target the production of bad trees. I then discuss the robustness of the result, and I suggest directions for future research.

For simplicity, I will consider an economy in which there is no aggregate risk; that is, I assume that $P(\alpha = \overline{\alpha}) = 1$ for some $\overline{\alpha} \in [0, 1)$.[41] Suppose that besides being able to choose to issue government bonds, B, and implement a transaction wedge, ω, as in Section 4, the planner can choose the maximum number of trees produced, H. As in the proof of Proposition 8, I assume that the planner chooses P_S and $\overline{T}_1$; we can then recover $\{B, \omega\}$ by setting $\omega = \frac{P_S}{P_B} - 1$ and $B = \mu_B \overline{T}_1$. Then, the planner's problem becomes

$$\max_{P_S, \overline{T}_1, H} \int_{\mu_B}^{\mu^{\max}} \mu \left[W_1 + \overline{T}_1\right] dG(\mu) + \left[\int_{\widetilde{\mu}_B}^{\mu^{\max}} \mu H_B dG(\mu) + \int_{\mu_S}^{\mu^{\max}} \mu H_G dG(\mu)\right] P_S + ZH_G + \alpha Z H_B - \frac{\chi}{1+\chi}\mathbb{T}_2$$

subject to

$$\frac{\gamma_G}{\gamma_B} \geq C'(H_G)$$
$$H_B \leq W_0 - C(H_G)$$
$$H_G + H_B \leq H$$
$$0 \leq \mathbb{T}_2(\alpha) \leq W_2$$

where

$$\mathbb{T}_2 = (1+\chi)\,\mu_B\left[\overline{T}_1 + (P_S - P_B)\,S\right].$$

The next proposition shows that the planner does not find it optimal to limit the production of trees, even if this would only reduce the production of bad trees at the margin.

Proposition 10 (Optimality of Bad Trees)**.** *Consider the economy in Section 2 a degenerate distribution for α, that is, Prob($\alpha = \overline{\alpha}$) $= 1$ for some $\overline{\alpha} \in [0, 1)$. If the optimal policy features $B > 0$, then it is weakly optimal to set $H_B = W_0 - C(H_G)$, and it is strictly optimal if $\alpha > 0$.*

Proof. I show that independently of the choice of $\{B, \omega\}$, it is never optimal to restrict the production of trees.

First, note that the planner would never choose to leave the constraint $\frac{\gamma_G}{\gamma_B} = C'(H_G)$ slack. In that case, increasing H or P_S increases welfare, as the additional production of trees would be limited to good trees. Then, the planner's problem can be rewritten as

$$\max_{P_S, \overline{T}_1, H_B} \int_{\mu_B}^{\mu^{\max}} \mu \left[W_1 + \overline{T}_1\right] dG(\mu) + \left[\int_{\max\left\{\mu_B, \frac{\overline{\alpha} Z}{P_S}\right\}}^{\mu^{\max}} \mu H_B dG(\mu) + \int_{\mu_S}^{\mu^{\max}} \mu H_G dG(\mu)\right] P_S + ZH_G + \overline{\alpha} Z H_B - \frac{\chi}{1+\chi}\mathbb{T}_2$$

subject to

$$\frac{\gamma_G}{\gamma_B} = C'(H_G)$$
$$H_B \leq W_0 - C(H_G)$$
$$0 \leq \mathbb{T}_2(\alpha) \leq W_2$$

[41] When the distribution of α is degenerate, the existence of a competitive equilibrium is not guaranteed. However, this has no impact on the analysis of the optimal policy, which is the focus here.

and

$$\mathbb{T}_2 = (1+\chi)\left[\mu_B \overline{T}_1 + (\mu_B P_S S - \mu_B P_B S)\right]$$

We need to consider two cases:

i. If $P_S \geq \frac{\overline{\alpha} Z}{\mu_B}$, $\tilde{\mu}_B = \mu_B$. Then, we can rewrite the planner's objective as

$$\int_{\mu_B}^{\mu^{\max}} \mu\left[W_1 + H_B P_S + \overline{T}_1\right] dG(\mu) + \int_{\mu_S}^{\mu^{\max}} \mu H_G P_S dG(\mu) + Z H_G + \alpha Z H_B - \frac{\chi}{1+\chi}\mathbb{T}_2$$

Moreover, the market clearing condition for trees is

$$G(\mu_B) W_1 = [1 - G(\mu_S)] H_G P_S + [1 - G(\mu_B)]\left[H_B P_S + \overline{T}_1\right]$$

Finally, the transfers in period 2 are

$$\mathbb{T}_2 = (1+\chi)\left[\mu_B\left[H_B P_S + \overline{T}_1\right] + [\mu_B P_S - Z][1 - G(\mu_S)] H_G - \overline{\alpha} Z H_B\right]$$

For any level of P_S, if $\overline{\alpha} = 0$, $\overline{T}_1$ and H_B become equivalent for the planner. When $\overline{\alpha} > 0$, H_B dominates $\overline{T}_1$, as it reduces the cost of the intervention.

ii. If $P_S < \frac{\overline{\alpha} Z}{\mu_B}$, $\tilde{\mu}_B = \frac{\overline{\alpha} Z}{P_S}$. Then, we can rewrite the planner's objective as

$$\int_{\mu_B}^{\mu^{\max}} \mu\left[W_1 + \overline{T}_1\right] dG(\mu) + \left[\int_{\frac{\overline{\alpha} Z}{P_S}}^{\mu^{\max}} \mu H_B dG(\mu) + \int_{\mu_S}^{\mu^{\max}} \mu H_G dG(\mu)\right] P_S + Z H_G + \overline{\alpha} Z H_B - \frac{\chi}{1+\chi}\mathbb{T}_2$$

Moreover, the market clearing condition for trees is

$$G(\mu_B) W_1 = [1 - G(\mu_S)] H_G P_S + [1 - G(\mu_B)] \overline{T}_1 + \left[1 - G\left(\frac{\overline{\alpha} Z}{P_S}\right)\right] H_B P_S$$

Finally, the transfers in period 2 are

$$\mathbb{T}_2 = (1+\chi)\left[\mu_B\left[[1 - G(\mu_B)]\overline{T}_1 + \left[1 - G\left(\frac{\overline{\alpha} Z}{P_S}\right)\right] H_B P_S\right] + [\mu_B P_S - Z][1 - G(\mu_S)] H_G - \left[1 - G\left(\frac{\overline{\alpha} Z}{P_S}\right)\right] H_B \overline{\alpha} Z + \mu_B G(\mu_B) \overline{T}_1\right]$$

Then, in this case H_B always dominates $\overline{T}_1$.

□

Proposition 10 states that it is never strictly optimal for the planner to restrict the production of bad trees. To understand this result, one should note that in this economy bad trees reduce welfare relative to the first best only through two channels: (i) by crowding out the production of good trees, and (ii) by distorting the decision to sell good trees. Since production decisions are unobservable, the planner cannot directly affect (i); production decisions are determined by (14). To deal with (ii), the planner can adjust the market wedge and transfers to neutralize the effect of bad trees on the market. The case with $\alpha = 0$ shows this in a transparent way. In this case, H_B and transfers are equivalent. The reason is that as long as P_S does not change, higher H_B increases the volume traded in the market and allows agents with $\mu \geq \mu_B$ to consume more. This is exactly what can be achieved with transfers. Moreover, the cost of having more bad trees is the transaction subsidy, which has to compensate for the bad trees in the market. But this subsidy is just "paying" for the bad trees, which is effectively the cost the planner faces with the transfers. Moreover, as long as P_S does not change, the incentives to produce tree quality is not affected. Finally, as α increases, bad trees become better than transfers.

It should be noted that the result in Proposition 10 is highly sensitive to the assumptions in the model. In particular, it relies on the fact that there is no alternative use for the endowment in period 0, so there is actually no opportunity cost of producing bad trees except for their impact on the economy in period 1. If agents could also consume in period 0, the result would depend on the comparison between the utility of consumption in period 0 with the liquidity benefits of bad trees in period 1 and the consumption of fruit in period 2. For example, if α is sufficiently close to zero, it would be optimal to restrict the production of trees. The planner could achieve this by restricting the *total* production of trees since the production of bad trees is the residual after the desired amount of good trees has been produced (interestingly, such a policy would need to be accompanied by transaction taxes to discourage the production of bad trees). Thus, the trade-off would be between the liquidity needs of the economy, the quality of bad trees, the cost of intervention, and the alternative uses in period 0.

Moreover, this intervention depends on the production technologies in period 0. For example, consider an alternative technology by which a fraction $q(e)$ of all the trees produced are good while $1 - p(e)$ are bad, where e is costly effort and $q(e) \in (0, 1)$ with $q'(e) > 0$. In that case, restricting the number of trees produced might not lead to an improvement in the quality of trees in the economy.

These results suggest that more work is required to understand the role of macroprudential policy that limites ex-ante investment in economies with moral hazard and adverse selection.